FACTORIZATIONS OF INFINITELY DIFFERENTIABLE FUNCTIONS AND SMOOTH VECTORS

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ABSTRACT. Let G be a real Lie group. Let $\mathcal{D}(G)$ be the set of compactly supported, infinitely differentiable functions on G. We show that if $f \in \mathcal{D}(G)$, then f is a finite sum of functions of the form $g * h$, where $g \in \mathcal{D}(G), h \in \mathcal{D}(G)$. Question: can f be written as $g * h$, where $g \in \mathcal{D}(G), h \in \mathcal{D}(G)$? Answer: yes, for a large class of groups (including for example the semi-simple groups with finite center), no for $G = \mathbf{R}^2$.

Let E be a Fréchet space, π a continuous representation of G on E. We show that every smooth vector for π belongs to the Gårding space.

1. INTRODUCTION

Let G be a (real) Lie group, f an element of $\mathcal{D}(G)$, i.e. an infinitely differentiable complex valued function on G with compact support. For any integer $n > 0$, we have that f is a finite sum of functions of the form $g * h$, where $g \in \mathcal{D}(G)$ and h is n times differentiable with compact support([3], p. 199; [1], p. 251; [4], p.23). In fact, we show that f is a finite sum of functions of the form $g * h$, where $g \in \mathcal{D}(G), h \in \mathcal{D}(G)$ (th. 3.1). For $G = \mathbf{R}^n$, this result was established in [12].

Let E be a Fréchet space, π a continuous representation of G on E, E_∞ the set of smooth vectors of E for π. To show that E_∞ is dense in E, one introduces classically the Gårding space E^∞ of E, the set of linear combinations of vectors of the form $\pi(f)\xi$ where $f \in \mathcal{D}(G)$ and $\xi \in E$. In fact, we prove that $E_\infty = E^\infty$ (th. 3.3).

These results can be qualified as theorems of "weak factorization". One wonders if there exists a "strong factorization", i.e. every element of $\mathcal{D}(G)$ is of the form $g * h$, where $g \in \mathcal{D}(G), h \in \mathcal{D}(G)$. The question, for $G = \mathbf{R}$, was posed by L. Ehrenpreis ([7], p. 584). A negative answer for $G = \mathbf{R}^3$ was given by L. Rubel, W. Squires and B. Taylor [12]. We will see that the answer is positive for a large class of groups containing for example the semi-simple groups with finite center (th. 4.9), and that the answer is negative for $G = \mathbf{R}^2$ and hence for all G which admit $\mathbf{R}^2$ as a quotient (th. 6.1 and 6.3). The groups which form the main obstacle to a general solution of the strong factorization problem are $\mathbf{R}$ and the universal covering of $SL(2, \mathbf{R})$. We will also obtain a strong factorization result for smooth vectors (th. 4.11).

We will establish variants of the preceding results for simply-connected nilpotent G (th. 7.1, 7.3, 7.4). These variants will be used to define a unitary representation of G on the space of rapidly decaying distributions on G (cor. 7.5). These distributions were considered recently ([9], [10]), but the corresponding operators have not been defined for central distributions and certain representations.

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Notation. We use the notation of L. Schwartz, $\mathcal{D}, \mathcal{D}^k, \mathcal{E}, \mathcal{D}', \mathcal{E}'$. For example, $\mathcal{D}^k(\mathbf{R}^m)$ is the set of complex valued functions on $\mathbf{R}^m$ which are k times continuously differentiable with compact support, and $\mathcal{D}(\mathbf{R}^m) = \mathcal{D}^\infty(\mathbf{R}^m)$. If $\alpha \in \mathbf{N}^m$, we denote by D^α the corresponding partial differentiation operator on $\mathbf{R}^m$. If $f \in \mathcal{D}^k(\mathbf{R}^m)$, we let $\|f\|_k = \sum_{0\leq|\alpha|\leq k} \sup |D^\alpha f|$.

If $T \in \mathcal{E}'(\mathbf{R}^m)$, we denote by supp (T) the support of T, and by co (T) the convex hull of supp (T).

If X is a topological space and $A \subset X$, we denote by $\mathrm{adh}_X\, A$ the closure of A in X.

If x is a point in a locally compact space, we denote by δ_x the Dirac measure at x. We denote by δ the Dirac measure at the origin in $\mathbf{R}$, and by $\delta', \ldots, \delta^{(n)}, \ldots$ its successive derivatives.

If G is a nilpotent, simply-connected Lie group, we identify it with its Lie algebra by the exponential map, and hence one can define $\mathcal{S}(G)$, $\mathcal{S}'(G)$, $\mathcal{O}'_c(G)$ (always with the notation of L. Schwartz); eventually we will also consider $\mathcal{O}_c(G)$ (cf. [8], chap. II, p. 131 for the definition of $\mathcal{O}_c$ for $\mathbf{R}^m$). We denote by $\mathcal{S}(\mathbf{Z})$ the space of sequences of complex numbers indexed by $\mathbf{Z}$ which are of rapid decay.

We denote by e the identity element of a group.

2. Construction of certain auxiliary functions

2.1. Until 2.4 , we fix a strictly increasing subsequence

$$\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_k, \ldots)$$

of the sequence $(1, 2, \ldots, 2^k, \ldots)$. For an integer $j \geq 0$, one has

$$(1) \qquad \begin{aligned} \textstyle\prod_{k>j} \left(1 - \frac{\lambda_j^2}{\lambda_k^2}\right) &\geq \left(1 - \tfrac{1}{2^2}\right)\left(1 - \tfrac{1}{4^2}\right)\left(1 - \tfrac{1}{8^2}\right)\cdots \\ &\geq \exp\left(-2\left(\tfrac{1}{2^2} + \tfrac{1}{4^2} + \cdots\right)\right) = e^{-2/3} \geq \tfrac{1}{2}, \end{aligned}$$

$$\left|1 - \frac{\lambda_j^2}{\lambda_k^2}\right| \geq 3 \qquad \text{if } k \leq j-1. \tag{2}$$

2.2. For $x \in \mathbf{R}$, let

$$\varphi_\lambda(x) = \prod_{k=0}^{\infty} \left(1 + \frac{x^2}{\lambda_k^2}\right), \qquad \chi_\lambda(x) = \varphi_\lambda(x)^{-1}.$$

The function φ_λ is even and extends to an entire function on $\mathbf{C}$, again denoted by φ_λ; its zeros are at $\pm i\lambda_j$ and they are simple. The function χ_λ is even and extends to a meromorphic function on $\mathbf{C}$, again denoted by χ_λ; its poles are at $\pm i\lambda_j$ and they are simple; the residue of χ_λ at $i\lambda_j$ is

$$\frac{1}{\varphi'_\lambda(i\lambda_j)} = \frac{1}{2i}\,\lambda_j \prod_{k\neq j}\left(1 - \frac{\lambda_j^2}{\lambda_k^2}\right)^{-1}. \tag{3}$$

By (1), (2), (3), one has

$$\frac{1}{|\varphi'_\lambda(i\lambda_j)|} \leq \frac{1}{2}\,|\lambda_j|\,2{\cdot}3^{-j} \leq |\lambda_j|. \tag{4}$$

An elementary calculation shows that, for $x, t \in \mathbf{R}$, one has

$$\left|1 + \frac{(x+it)^2}{\lambda_k^2}\right|^2 \geq \left(1 - \frac{t^2}{\lambda_k^2}\right)^2 + \frac{x^4}{\lambda_k^4}. \tag{5}$$

Let $t = t_j = (\lambda_j \lambda_{j+1})^{1/2}$. Then

$$\prod_{k\geq j+2}\left(1 - \frac{t_j^2}{\lambda_k^2}\right) \geq \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{8^2}\right)\cdots \geq \frac{1}{2},$$

$$1 - \frac{t_j^2}{\lambda_{j+1}^2} = 1 - \frac{\lambda_j}{\lambda_{j+1}} \geq \frac{1}{2},$$

$$\left|1 - \frac{t_j^2}{\lambda_k^2}\right| \geq 1 \qquad \text{if } k \leq j,$$

hence

$$|\varphi_\lambda(x+it_j)| \geq \frac{1}{4}\left(1 + \frac{x^4}{\lambda_0^2}\right)^{1/2}, \qquad |\chi_\lambda(x+it_j)| \leq 4\left(1 + \frac{x^4}{\lambda_0^2}\right)^{-1/2}. \tag{6}$$

2.3. The function φ_λ on $\mathbf{R}$ increases faster than any polynomial at infinity, hence χ_λ is of rapid decay. For $y \in \mathbf{R}$, let

$$\psi_\lambda(y) = \int_{-\infty}^{\infty} e^{-2i\pi xy}\chi_\lambda(x)dx = \int_{-\infty}^{\infty} e^{2i\pi xy}\chi_\lambda(x)dx.$$

Then ψ_λ is even and infinitely differentiable on $\mathbf{R}$. Let $t \in (\lambda_k, \lambda_{k+1})$. By (5) and (6), the calculation of residues gives

$$\int_{-\infty}^{\infty} e^{2i\pi xy}\chi_\lambda(x)dx - \int_{-\infty}^{\infty} e^{2i\pi(x+it)y}\chi_\lambda(x+it)dx = \sum_{j=0}^{k} \frac{1}{\varphi'(i\lambda_j)}e^{-2\pi\lambda_j y},$$

i.e.

$$\psi_\lambda(y) = e^{-2\pi ty}\int_{-\infty}^{\infty} e^{2i\pi xy}\chi_\lambda(x+it)dx + \sum_{j=0}^{k} \frac{1}{\varphi'(i\lambda_j)}e^{-2\pi\lambda_j y}.$$

Suppose $y > 0$. Let $t = t_k = (\lambda_k\lambda_{k+1})^{1/2}$, and let k approach ∞. By (6), one obtains

$$\psi_\lambda(y) = \sum_{j=0}^{\infty} \frac{1}{\varphi'(i\lambda_j)}e^{-2\pi\lambda_j y} \qquad \text{for } y > 0.$$

Formally, one deduces

$$y^m\frac{d^n\psi_\lambda}{dy^n} = (-2\pi)^n\sum_{j=0}^{\infty} \frac{1}{\varphi'(i\lambda_j)}\lambda_j^n y^m e^{-2\pi\lambda_j y} \qquad (y > 0). \tag{7}$$

The maximum of $y^m e^{-2\pi\lambda_j y}$ for $y > 0$ is attained when $y = m/2\pi\lambda_j$, and it is equal to $(m/e)^m(2\pi\lambda_j)^{-m}$. However, for $m > n+1$, one has

$$\sum_{j=0}^{\infty}\left|\frac{\lambda_j^{n-m}}{\varphi'(i\lambda_j)}\right| < \infty$$

by (4). Hence, if $m > n+1$, the series (7) converges uniformly for $y > 0$ and it gives the value of $y^m\frac{d^n\psi_\lambda}{dy^n}$, where $y^m\frac{d^n\psi_\lambda}{dy^n} \to 0$ at infinity.

Therefore, one has $\varphi_\lambda \in \mathcal{S}(\mathbf{R})$ and consequently $\chi_\lambda \in \mathcal{S}(\mathbf{R})$.

2.4. Recall the equality

$$\frac{d^n\psi_\lambda}{dy^n} = (-2\pi)^n\sum_{j=0}^{\infty} \frac{\lambda_j^n}{\varphi'(i\lambda_j)}e^{-2\pi\lambda_j y},$$

where the series converges uniformly for $y \geq y_0 > 0$ by (4). One then deduces

$$\begin{aligned} \sup_{y\geq 1}\left|\tfrac{d^n\psi_\lambda}{dy^n}\right| &\leq (2\pi)^n\textstyle\sum_{j=0}^{\infty}\lambda_j^{n+1}e^{-2\pi\lambda_j} \\ &\leq (2\pi)^n\textstyle\sum_{j=0}^{\infty}2^{(n+1)j}e^{-2\pi 2^j}. \end{aligned} \tag{8}$$

It is important to note that this last expression is independent of the choice of the sequence λ.

2.5. Lemma: *Let $(\beta_0, \beta_1, \beta_2, \ldots)$ be a sequence of positive numbers. Then there exists a sequence of positive numbers $(\alpha_0, \alpha_1, \alpha_2, \ldots)$ and functions $f \in \mathcal{S}(\mathbf{R})$, $g \in \mathcal{D}(\mathbf{R})$, $h \in \mathcal{D}(\mathbf{R})$ such that*

(1) $\alpha_n \leq \beta_n$ for $n \geq 1$,

(2) $\sum_{n=0}^{p}(-1)^n \alpha_n \delta^{(2n)} * f \to \delta$ in $\mathcal{S}'(\mathbf{R})$ as $p \to \infty$,

(3) $\sum_{n=0}^{p}(-1)^n \alpha_n \delta^{(2n)} * g \to \delta + h$ in $\mathcal{E}'(\mathbf{R})$ as $p \to \infty$.

We continue to use the notation in 2.1-2.4. Fix a function $\omega \in \mathcal{D}(\mathbf{R})$ such that ω is even, equal to 1 in $[-2, 2]$ and supported in $[-3, 3]$. Let $\omega_\lambda = \psi_\lambda \cdot \omega$.

By (8), there exists a sequence $(P_0, P_1, \ldots)$ of positive numbers such that for any sequence λ one has

$$\sup_{y \geq 1} \left| \frac{d^n \omega_\lambda}{dy^n} \right| \leq P_n.$$

Suppose $\lambda_0, \lambda_1, \ldots, \lambda_{q-1}$ are chosen such that in the expansion

$$\sum_{n \geq 0} \gamma_n x^{2n} \text{ of } \left(1 + \frac{x^2}{\lambda_0^2}\right) \cdots \left(1 + \frac{x^2}{\lambda_{q-1}^2}\right)$$

one has

$$\gamma_n < \inf \left(\beta_n, \frac{1}{n^2 P_{2n}}, \frac{1}{n^2 P_{2n+1}}, \ldots, \frac{1}{n^2 P_{2n+n}} \right)$$

for $n = 1, \ldots, q$. Then, one can choose λ_q such that in the expansion

$$\sum_{n \geq 0} \gamma'_n x^{2n} \text{ of } \left(1 + \frac{x^2}{\lambda_0^2}\right) \cdots \left(1 + \frac{x^2}{\lambda_q^2}\right)$$

one has

$$\gamma'_n < \inf \left(\beta_n, \frac{1}{n^2 P_{2n}}, \frac{1}{n^2 P_{2n+1}}, \ldots, \frac{1}{n^2 P_{2n+n}} \right)$$

for $n = 1, \ldots, q+1$. Continuing this way, one obtains a sequence λ such that in the expansion $\sum_{n \geq 0} \alpha_n x^{2n}$ of φ_λ one has

$$\alpha_n \leq \inf \left(\beta_n, \frac{1}{n^2 P_{2n}}, \frac{1}{n^2 P_{2n+1}}, \ldots, \frac{1}{n^2 P_{2n+n}} \right) \qquad \text{for } n \geq 1.$$

For all $x \in \mathbf{R}$, one has $0 \leq (\sum_{n=0}^{p} \alpha_n x^{2n}) \chi_\lambda(x) \leq 1$, and

$$(\sum_{n=0}^{p} \alpha_n x^{2n}) \chi_\lambda(x) \to 1$$

as $p \to \infty$. Hence $(\sum_{n=0}^{p} \alpha_n x^{2n}) \chi_\lambda(x) \to 1$ in $\mathcal{S}'(\mathbf{R})$ as $p \to \infty$. Similarly,

$$\sum_{n=0}^{p} (-1)^n \alpha_n \frac{\delta^{(2n)}}{(2\pi)^{2n}} * \psi_\lambda \to \delta$$

in $\mathcal{S}'(\mathbf{R})$ as $p \to \infty$.

The proof will be finished by showing that

$$\theta_p = \sum_{n=0}^{p} (-1)^n \alpha_n \frac{\delta^{(2n)}}{(2\pi)^{2n}} * \omega_\lambda,$$

whose support is contained in $[-3,3]$, converges, in $\mathcal{E}'(\mathbf{R})$, to a distribution of the form $\delta + h$, where $h \in \mathcal{D}(\mathbf{R})$. It is enough to consider the restrictions of θ_p to $(-2,2)$, $(1,4)$, $(3,\infty)$. We have $\theta_p = 0$ on $(3,\infty)$, and

$$\theta_p|_{(-2,2)} = \left(\sum_{n=0}^{p} (-1)^n \alpha_n \frac{\delta^{(2n)}}{(2\pi)^{2n}} * \psi_\lambda \right) \Big|_{(-2,2)},$$

hence θ_p converges to δ in $\mathcal{D}'((-2,2))$. Finally, for $y \geq 1$, one has

$$\left| \alpha_n \frac{\delta^{(2n+p)}}{(2\pi)^{2n}} * \omega_\lambda(y) \right| \leq \alpha_n P_{2n+p} \leq \frac{1}{n^2} \qquad \text{if } n \geq p,$$

hence

$$\left(\sum_{n=0}^{p} (-1)^n \alpha_n \frac{\delta^{(2n)}}{(2\pi)^{2n}} * \omega_\lambda \right) \Big|_{(1,4)}$$

has a limit in $\mathcal{E}((1,4))$ as $p \to \infty$.

2.6. *Remark:* It is clear that, for any $\varepsilon > 0$, one can require functions g, h of Lemma 2.5 to have their supports contained in $[-\varepsilon, \varepsilon]$.

3. Weak factorization of infinitely differentiable functions and smooth vectors

3.1. Theorem: *Let G be a Lie group, V a neighborhood of e in G, and $\varphi \in \mathcal{D}(G)$. Then φ is a finite sum of functions of the form $\psi_1 * \psi_2$, where $\psi_1, \psi_2 \in \mathcal{D}(G), \operatorname{supp}(\psi_1) \subset V, \operatorname{supp}(\psi_2) \subset \operatorname{supp}(\varphi)$.*

(1) Let $\mathfrak{g}$ be the Lie algebra of G. One can choose a basis $(x_1, \ldots, x_m)$ of $\mathfrak{g}$ satisfying the following property: if ζ denotes the map

$$(t_1, \ldots, t_m) \mapsto (\exp t_1 x_1) \cdots (\exp t_m x_m)$$

from $\mathbf{R}^m$ to G, the restriction of ζ to $(-1,1)^m$ is a diffeomorphism of $(-1,1)^m$ onto an open set Ω of G.

Let β be a left Haar measure of G and β_Ω its restriction to Ω.

(2) Let $(u_1, u_2, \ldots)$ be a basis of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. If $u \in U(\mathfrak{g})$, u defines a right invariant differential operator D_u on G, and one has $u * \varphi = D_u(\varphi)$; that being recalled,

let

$$\sup_{s\in G} |(u_i * x_1^{2n} * \varphi)(s)| = M_{ni}. \tag{9}$$

Let $\varepsilon \in (0, 1/2)$. By 2.5 and 2.6, one can choose $\alpha_0, \alpha_1, \alpha_2, \ldots > 0$ and $g, h \in \mathcal{D}(\mathbf{R})$ whose supports are contained in $[-\varepsilon, \varepsilon]$ such that

$$\sum_{n=0}^{\infty} \alpha_n M_{ni} < \infty \quad \text{for all } i \tag{10}$$

and

$$\sum_{n=0}^{p} (-1)^n \alpha_n \delta^{(2n)} * g \to \delta + h \quad \text{in } \mathcal{E}'(\mathbf{R}) \text{ as } p \to \infty. \tag{11}$$

The map $t_1 \mapsto \exp t_1 x_1$ from $\mathbf{R}$ to G transforms the measures $g(t_1)dt_1, h(t_1)dt_1$ on $\mathbf{R}$ to measures μ, ν oni G. One obtains from (11) that

$$\mu * \sum_{n=0}^{p} (-1)^n \alpha_n x_1^{2n} = \sum_{n=0}^{p} (-1)^n \alpha_n x_1^{2n} * \mu \to \delta_e + \nu$$

in $\mathcal{E}'(G)$ as $p \to \infty$. Hence

$$\mu * \sum_{n=0}^{p} (-1)^n \alpha_n x_1^{2n} * \varphi \to \varphi + \mu * \varphi$$

in $\mathcal{E}'(G)$ as $p \to \infty$. Furthermore, because of (9) and (10), $\sum_{n=0}^{p} (-1)^n \alpha_n x_1^{2n} * \varphi$ converges in $\mathcal{D}(G)$ to an element $\psi \in \mathcal{D}(G)$. Then $\mu * \psi = \varphi + \nu * \varphi$. So, φ is a sum of functions of the form $\xi * \chi$, where $\chi \in \mathcal{D}(G)$, $\operatorname{supp}(\chi) \subset \operatorname{supp}(\varphi)$, and where ξ is the image under $t_1 \mapsto \exp t_1 x_1$ of a measure of the form $f(t_1)dt_1$ with $f \in \mathcal{D}(\mathbf{R})$ and $\operatorname{supp} f \subset [-\varepsilon, \varepsilon]$.

(3) Continuing in this way, one deduces from (2) that φ is a finite sum of functions of the form $\xi_1 * \xi_2 * \ldots * \xi_m * \chi$, where $\chi \in \mathcal{D}(G), \operatorname{supp}(\chi) \subset \operatorname{supp}(\varphi)$, and where ξ_i is the image under $t_i \mapsto \exp t_i x_i$ of a measure of the form $f_i(t_i)dt_i$, with $f_i \in \mathcal{D}(\mathbf{R})$ and $\operatorname{supp} f_i \subset [-\varepsilon, \varepsilon]$. But $\xi_1 * \xi_2 * \ldots * \xi_m$ is the image of $\xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_m$ under the product map $G \times \ldots \times G \to G$. Hence $\xi_1 * \xi_2 * \ldots * \xi_m$ is the image under ζ of the measure

$$f_1(t_1) \cdots f_m(t_m) dt_1 \cdots dt_m$$

on $\mathbf{R}^m$. The function $(t_1, \ldots, t_m) \mapsto f_1(t_1) \cdots f_m(t_m)$ belongs to $\mathcal{D}(\mathbf{R}^m)$ and its support is contained in $[-\varepsilon, \varepsilon]^m$.

The image of the restriction of $dt_1 \cdots dt_m$ to $(-1, 1)^m$ under ζ is the product of β_Ω with a function in $\mathcal{E}(\Omega)$. Hence $\xi_1 * \ldots * \xi_m$ is the product of β with a function in $\mathcal{D}(G)$. If ε is small enough, this function of $\mathcal{D}(G)$ will have its support contained in V.

3.2. If H is a Hilbert space, and $p \in (0, \infty)$, let $\mathcal{L}(H)$ denote the Banach space of continuous endomorphisms of H, and $\mathcal{L}^p(H)$ denote the Banach space of compact elements $T \in \mathcal{L}(H)$ such that $\sum_n \lambda_n^{p/2} < \infty$, where (λ_n) is the sequence of eigenvalues of $T * T$ (counted with multiplicity). For example, $\mathcal{L}^2(H)$ (resp. $\mathcal{L}^1(H)$) is the set of Hilbert-Schmidt operators (resp. trace class operators) on H.

Corollary: *Let G be a Lie group, H a Hilbert space, π a continuous unitary representation of G on H. Assume that there exists a $p \in (0, \infty)$ such that $\pi(\varphi) \in \mathcal{L}^p(H)$ for every $\varphi \in \mathcal{D}(G)$. Then, for every $\varphi \in \mathcal{D}(G)$, the sequence of eigenvalues of $\pi(\varphi) * \pi(\varphi)$ in decreasing order (counted with multiplicity) is of rapid decay.*

Let $\varphi \in \mathcal{D}(G)$. By 3.1, for any integer $n > 0$, $\pi(\varphi)$ is a finite sum of products of the elements in $\mathcal{L}^p(H)$, hence $\pi(\varphi) \in \mathcal{L}^{p/n}(H)$ ([6], p. 1093, lemma 9c). This proves the corollary.

(With the preceding hypotheses, the linear form $\varphi \mapsto \operatorname{tr} \pi(\varphi)$ on $\mathcal{D}(G)$ is a distribution (the "character" of π); in fact, the map $\varphi \mapsto \pi(\varphi)$ is continuous from $\mathcal{D}(G)$ to $\mathcal{L}(H)$, and hence continuous from $\mathcal{D}(G)$ to $\mathcal{L}^1(H)$ by the closed graph theorem.)

3.3. Theorem: *Let G be a Lie group, V a neighborhood of e in G, E a Fréchet space, π a continuous representation of G on E, E_∞ the set of smooth vectors of E for π, and $\xi \in E_\infty$. Then ξ is a finite sum of vectors of the form $\pi(\varphi)\eta$ where $\varphi \in \mathcal{D}(G)$, $\operatorname{supp}(\varphi) \subset V$ and $\eta \in E_\infty$.*

We proceed as in the proof of 3.1. We continue to use the notation for $(x_1, \ldots, x_m)$, $(u_1, u_2, ldots)$ of 3.1. Let $(p_1, p_2, \ldots)$ be a sequence of semi-norms defining the topology of E. Let

$$p_j(\pi(u_i)\pi(x_1)^n\xi) = M_{nij}.$$

Let $\varepsilon \in (0, 1/2)$. Choose $\alpha_0, \alpha_1, \alpha_2, \ldots > 0$, $g, h \in \mathcal{D}(\mathbf{R})$ with supports contained in $[-\varepsilon, \varepsilon]$ such that

$$\sum_n \alpha_n M_{nij} < \infty \text{ for every } i, j,$$

$$\sum_{n=0}^{p} (-1)^n \alpha_n \delta^{(2n)} * g \to \delta + h \text{ in } \mathcal{E}'(\mathbf{R}) \text{ as } p \to \infty.$$

Define μ, ν as in 3.1. One has

$$\pi(\mu) \sum_{n=0}^{p} (-1)^n \alpha_n \pi(x_1)^{2n} \xi = \pi(\mu * \sum_{n=0}^{p} (-1)^n \alpha_n x_1^{2n})\, \xi \to \xi + \pi(\nu)\xi$$

in E with the weak topology. Furthermore, one has

$$\sum_{n=0}^{\infty} p_j(\pi(u_i)\alpha_n \pi(x_1)^{2n}\xi) < \infty,$$

for every i, j, hence $\sum_{n=0}^{p}(-1)^n\alpha_n\pi(x_1)^{2n}\xi$ converges in the Fréchet space E_∞ to an element η in E_∞. One then deduces that

$$\pi(\mu)\eta = \xi + \pi(\nu)\xi.$$

The proof is then achieved inductively as in 3.1.

4. Strong factorization of infinitely differentiable functions and smooth vectors

4.1. Lemma: *Let C be a closed subset of $\mathcal{S}(\mathbf{Z})$. Then there exists $(\delta_n)_{n\in\mathbf{Z}} \in \mathcal{S}(\mathbf{Z})$ such that*
(a) $\delta_n > 0$ for all n,
(b) for any $(\varepsilon_n)_{n\in\mathbf{Z}} \in C$, one has $|\varepsilon_n| \leq \delta_n$ for every n.

For $p \in \mathbf{Z}$, let

$$\delta_p = \sup_{(\varepsilon_n)\in C} |\varepsilon_p|.$$

If k is a positive integer, one has

$$\sup_{(\varepsilon_n)\in C}\ \sup_{p\in\mathbf{Z}} (|\varepsilon_p|(1+|p|^k)) < \infty;$$

hence

$$\sup_{p\in\mathbf{Z}} \delta_p(1+|p|^k) < \infty,$$

which proves that $(\delta_n) \in \mathcal{S}(\mathbf{Z})$. The condition (b) is easily verified. By a slight modification of (δ_n), one can show that condition (a) also holds.

4.2. Lemma: *Let U be an open subset of $\mathbf{R}^m$, φ an infinitely differentiable map with compact support from U to $\mathcal{S}(\mathbf{Z})$. For any $u \in U$, let $\varphi(u) = (\varphi_n(u))_{n\in\mathbf{Z}}$. Then*
(i) there exists $\beta = (\beta_n)_{n\in\mathbf{Z}} \in \mathcal{S}(\mathbf{Z})$ such that $\beta_n > 0$ for all n, and that, for any $\alpha \in \mathbf{N}^m$, one has

$$\sup_{u\in U, n\in\mathbf{Z}} |D^\alpha\varphi_n(u)|\beta_n^{-2} < \infty\ ,$$

(ii) suppose $\beta = (\beta_n)_{n\in\mathbf{Z}}$ satisfies the properties in (i) and $\gamma = (\gamma_n)_{n\in\mathbf{Z}} \in \mathcal{S}(\mathbf{Z})$ is such that $\gamma_n \geq \beta_n$ for all n. For all $n \in \mathbf{Z}$ and $u \in U$, let $\psi_n(u) = \gamma_n^{-1}\varphi_n(u)$. One has $\psi(u) = (\psi_n(u))_{n\in\mathbf{Z}} \in \mathcal{S}(\mathbf{Z})$, $\varphi(u) = \gamma\psi(u)$ for all $u \in U$ and ψ is an infinitely differentiable map from U to $\mathcal{S}(\mathbf{Z})$.

(i) For any $\alpha \in \mathbf{N}^m$, the image I_α of U in $\mathcal{S}(\mathbf{Z})$ under $D^\alpha\varphi$ is compact. Let $(\lambda_\alpha)_{\alpha\in\mathbf{N}^m}$ be a family of positive numbers (which exists) such that the union C of $\lambda_\alpha I_\alpha$ is closed in $\mathcal{S}(\mathbf{Z})$.

Lemma 4.1 then provides us with $(\delta_n)_{n\in\mathbf{Z}} \in \mathcal{S}(Z)$. If we let $\beta_n = \delta_n^{1/2}$ for $n \in \mathbf{Z}$, then property (i) is satisfied.

(ii) Let $\beta = (\beta_n)$, $\gamma = (\gamma_n)$, $\psi(u) = (\psi_n(u))$ be as in (ii). One has

$$|\psi_n(u)| \leq \beta_n^{-1}|\varphi_n(u)| \leq c\,\beta_n^{-1}\beta_n^2 = c\beta_n \ , \tag{12}$$

where c is independent of u and n; hence $\psi(u) \in \mathcal{S}(\mathbf{Z})$. It is clear that $\varphi(u) = \gamma\varphi(u)$ and that ψ has compact support. We now show that ψ is infinitely differentiable. We equip $\mathcal{S}(\mathbf{Z})$ not only with the strong topology but also with the weak topology defined by the dual space $\mathcal{S}'(\mathbf{Z})$ of the slowly increasing sequences; if $\omega = (\omega_n)_{n\in\mathbf{Z}} \in \mathcal{S}'(\mathbf{Z})$, one has

$$\langle\psi(u), \omega\rangle = \sum_{n\in\mathbf{Z}} \psi_n(u)\omega_n \ .$$

Let $\alpha \in \mathbf{N}^m$. Then

$$|D^\alpha\psi_n(u)| \leq c\beta_n \ , \tag{13}$$

where c is independent of u and of n (this is proven as in (12)). Since $\sum_{n\in\mathbf{Z}}\beta_n|\omega_n| < \infty$, $D^\alpha\langle\psi(u),\omega\rangle$ exists and is equal to $\sum_{n\in\mathbf{Z}} D^\alpha\psi_n(u)\omega_n$. Hence $D^\alpha\psi$ exists when ψ is considered with values in weak $\mathcal{S}(\mathbf{Z})$. Moreover, $D^\alpha\psi(u) = (D^\alpha\psi_n(u))_{n\in\mathbf{Z}}$. Each $D^\alpha\psi_n$ is a continuous map from U to $\mathcal{S}(\mathbf{Z})$ and as a result of (13), $D^\alpha\psi$ is a continuous map from U to strong $\mathcal{S}(\mathbf{Z})$. Hence ψ, considered as a map from U to strong $\mathcal{S}(\mathbf{Z})$, is infinitely differentiable ([3], 2.6.1).

4.3. Lemma: *Let U be an open subset of $\mathbf{R}^m$, φ an infinitely differentiable map with compact support from U to $\mathcal{D}(\mathbf{T})$. Then*
(i) there exists $\beta = (\beta_n)_{n\in\mathbf{Z}} \in \mathcal{S}(\mathbf{Z})$ such that $\beta_n > 0$ for all n and that, for all $\alpha \in \mathbf{N}^m$, the Fourier coefficients $\lambda_{\alpha n}(u)$ of $D^\alpha\varphi(u)$ satisfy

$$\sup_{u\in U, n\in\mathbf{Z}} |\lambda_{\alpha n}(u)|\beta_n^{-2} < \infty$$

(ii) let $\beta = (\beta_n)_{n\in\mathbf{Z}}$ be as in (i). Let

$$\gamma = (\gamma_n)_{n\in\mathbf{Z}} \in \mathcal{S}(\mathbf{Z})$$

be such that $\gamma_n \geq \beta_n$ for all n. Let χ be the element of $\mathcal{D}(\mathbf{T})$ whose Fourier coefficients are γ_n. Then there exists an infinitely differentiable map ψ from U to $\mathcal{D}(\mathbf{T})$ with compact support such that $\varphi(u) = \chi * \psi(u)$ for all $u \in U$.

This lemma follows from lemma 4.2 by an application of the Fourier transform.

4.4. Lemma: *Let P be a smooth principal $\mathbf{T}$-bundle, with the group $\mathbf{T}$ acting on the left on P. Let $\varphi \in \mathcal{D}(P)$. Then there exists $(\beta_n)_{n\in\mathbf{Z}} \in \mathcal{S}(\mathbf{Z})$ such that $\beta_n > 0$ for all n, and satisfying the following property:*
if $(\gamma_n)_{n\in\mathbf{Z}} \in \mathcal{S}(\mathbf{Z})$ satisfies $\gamma_n \geq \beta_n$ for all n and if χ is the element of $\mathcal{D}(\mathbf{T})$ whose Fourier coefficients are γ_n, then there exists $\psi \in \mathcal{D}(P)$ such that $\varphi = \chi * \psi$.

(a) Suppose that the fiber P is trivializable and that its basis is an open subset U of the space $\mathbf{R}^m$. Then P can be identified with $\mathbf{T} \times U$ and φ can be identified with an infinitely differentiable map with compact support from U to $\mathcal{D}(\mathbf{T})$. It suffices to apply lemma 4.3 to φ.

(b) Now consider the general case. Let $B = P/\mathbf{T}$ be the basis of P and $\pi : P \to B$ be the canonical map. There exist open sets $B_1, \ldots, B_q$ of B with the following properties: (1) each B_i is diffeomorphic to an open subset of the space $\mathbf{R}^{m_i}$, (2) each $\pi^{-1}(B_i)$ is trivializable, (3) $\operatorname{supp} \varphi \subset \pi^{-1}(B_1) \cup \ldots \cup \pi^{-1}(B_q)$. Then $\varphi = \varphi_1 + \ldots + \varphi_q$ with

$$\varphi_1 \in \mathcal{D}(\pi^{-1}(B_1)) \subset \mathcal{D}(P), \ldots, \varphi_q \in \mathcal{D}(\pi^{-1}(B_q)) \subset \mathcal{D}(P) .$$

Part (a) of the proof, applied to $\varphi_1, \ldots, \varphi_q$, produces q elements of $\mathcal{S}(\mathbf{Z})$. Let $(\beta_n)_{n\in\mathbf{Z}}$ be the sum of these q elements. Let (γ_n) and χ be as in the statement of the lemma. Then there exist $\psi_1 \in \mathcal{D}(\pi^{-1}(B_1)), \ldots, \psi_q \in \mathcal{D}(\pi^{-1}(B_q))$ such that $\varphi_1 = \chi * \psi_1, \ldots, \varphi_q = \chi * \psi_q$, and hence $\varphi = \chi * (\psi_1 + \ldots + \psi_q)$.

4.5. Lemma: *Let $(\beta_n)_{n\in\mathbf{Z}} \in \mathcal{S}(\mathbf{Z})$ be such that $\beta_n \geq 0$ for all n. Let V be a neighborhood of 0 in $\mathbf{T}$. Then there exists $\varphi \in \mathcal{D}(\mathbf{T})$ satisfying the following properties:*
(a) $\operatorname{supp} \varphi \subset V$,
(b) let $(\gamma_n)_{n\in\mathbf{Z}}$ be the Fourier coefficients of φ; then $\gamma_n \geq \beta_n$ for all n.

Let W be a closed symmetric neighborhood of 0 in $\mathbf{T}$ such that $W + W \subset V$. Let ψ be the element of $\mathcal{D}(\mathbf{T})$ whose Fourier coefficients are $\beta_n^{1/2}$. One can write ψ as a sum $\psi_1 + \ldots + \psi_p$ where, for every i, ψ_i is an element of $\mathcal{D}(\mathbf{T})$ whose support is contained in a translate of W. Let $(\beta_{in})_{n\in\mathbf{Z}}$ be the sequence of Fourier coefficients of ψ_i. Put $\omega_i = \psi_i * \tilde{\psi}_i$ (where $\tilde{\psi}_i(t) = \overline{\psi_i(-t)}$ for all $t \in \mathbf{T}$). Then $\operatorname{supp} \omega_i \subset W + W \subset V$. The Fourier coefficients of $\omega_1 + \ldots + \omega_p$ are the numbers

$$\delta_n = |\beta_{1n}|^2 + \cdots + |\beta_{pn}|^2 .$$

One has

$$\beta_n = (\beta_{1n} + \cdots + \beta_{pn})^2 \leq p\,(|\beta_{1n}|^2 + \cdots + |\beta_{pn}|^2) = p\,\delta_n$$

and it suffices to choose $\varphi = p\,(\omega_1 + \cdots + \omega_p)$.

4.6. Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra. An element x of $\mathfrak{g}$ is called *toroidal* if the one-parameter subgroup of G generated by x is closed and isomorphic to $\mathbf{T}$. (This definition depends not only on $\mathfrak{g}$ but also on G.) Let $\mathfrak{g}'$ be the vector subspace of $\mathfrak{g}$ generated by the toroidal elements of $\mathfrak{g}$; since $\mathfrak{g}'$ is stable under the adjoint representation of G, $\mathfrak{g}'$ is an *ideal* of $\mathfrak{g}$. The notations G, $\mathfrak{g}$, $\mathfrak{g}'$ are fixed until 4.8.

Let $\widetilde{SL}(2,\mathbf{R})$ denote the universal covering of $SL(2,\mathbf{R})$. If G is simple and is not isomorphic to $\widetilde{SL}(2,\mathbf{R})$, then the compact maximal subgroup of G is not trivial, hence $\mathfrak{g}' \neq 0$ and therefore $\mathfrak{g}' = \mathfrak{g}$.

4.7. Lemma: *If G is compact, then there exists a basis of $\mathfrak{g}$ consisting of toroidal elements.*

In fact, any element of $\mathfrak{g}$ generates a subgroup with a parameter whose closure is a torus $\mathbf{T}^n$, hence is the limit of toroidal elements in $\mathfrak{g}$.

4.8. Lemma: *Let L be a Levi subgroup of G. Suppose that: (1) $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$; (2) L is not contained in a distinguished subgroup isomorphic to $\widetilde{SL}(2,\mathbf{R})$.*

Then there exists a basis of $\mathfrak{g}$ consisting of toroidal elements.

Let $\mathfrak{l}$ be the Lie algebra of L, $\mathfrak{l} = \mathfrak{l}_1 \times \cdots \times \mathfrak{l}_{\mathfrak{p}} \times \mathfrak{m}_1 \times \cdots \times \mathfrak{m}_q$ be the decomposition of $\mathfrak{l}$ into simple ideals, where $\mathfrak{m}_i$ are isomorphic to $\mathfrak{sl}(2,\mathbf{R})$ and $\mathfrak{l}_i$ are not isomorphic to $\mathfrak{sl}(2,\mathbf{R})$. Let L_i, M_i be the analytic subgroups of G corresponding to $\mathfrak{l}_i$, $\mathfrak{m}_i$. By 4.6, each $\mathfrak{l}_i$ contains an element toroidal relative to L_i, and hence relative to G. By hypothesis (2) of the lemma, each M_i is a finite covering of $PSL(2,\mathbf{R})$; consequently, each $\mathfrak{m}_i$ contains an element toroidal relative to M_i, and hence relative to G. This thus proves that the ideal $\mathfrak{g}'$ of $\mathfrak{g}$ contains $\mathfrak{l}$. Therefore $\mathfrak{g}/\mathfrak{g}'$ is solvable. If $\mathfrak{g}/\mathfrak{g}'$ is non-zero, $\mathfrak{g}$ has an ideal $\mathfrak{g}'' \supset \mathfrak{g}'$ such that $\mathfrak{g}/\mathfrak{g}''$ is commutative and non-zero, which is a contradiction since $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$. Hence $\mathfrak{g} = \mathfrak{g}'$, which proves the lemma.

4.9. Theorem: *Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. Suppose that there exists a basis of $\mathfrak{g}$ consisting of toroidal elements (see 4.7 and 4.8 for examples of such groups).*

Let $\varphi \in \mathcal{D}(G)$, V a neighborhood of e in G. Then there exist $\psi_1, \psi_2 \in \mathcal{D}(G)$ such that $\varphi = \psi_1 * \psi_2$ and $\operatorname{supp}\varphi_1 \subset V$.

Let $(x_1,\ldots,x_m)$ be a basis of $\mathfrak{g}$ consisting of toroidal elements. Let ζ be the map

$$(t_1,\ldots,t_m) \mapsto (\exp t_1 x_1)\cdots(\exp t_m x_m)$$

of $\mathbf{R}^m$ to G. Let $\varepsilon > 0$ be such that the restriction of ζ to $(-\varepsilon,\varepsilon)^m$ is a diffeomorphism of $(-\varepsilon,\varepsilon)^m$ onto an open subset of G. Let $\varepsilon' \in (0,\varepsilon)$.

Let $T_1 = \exp \mathbf{R}x_1$, which is isomorphic to $\mathbf{T}$. Let dt_1 be a Haar measure on T_1. By 4.4 and 4.5, there exist $f_1 \in \mathcal{D}(T_1)$ and $\chi \in \mathcal{D}(G)$ such that supp $f_1 \subset \exp([-\varepsilon', \varepsilon']x_1)$ and $\varphi = (f_1 dt_1) * \chi$.

By induction, as in the proof of 3.1, one can deduce that $\varphi = \psi_1 * \psi_2$ where $\psi_1, \psi_2 \in \mathcal{D}(G)$ and where

$$\operatorname{supp} \psi_1 \subset \exp([-\varepsilon', \varepsilon']x_1) \cdots \exp([-\varepsilon', \varepsilon']x_m) \ ;$$

and consequently, supp $\psi_1 \subset V$ if ε' is small enough.

4.10. *Remark:* Let G and V be as in theorem 4.9. Let $\mathcal{K}$ be a compact subset of $\mathcal{D}(G)$. Then there exist $\psi_1 \in \mathcal{D}(G)$ and a compact subset $\mathcal{K}_2$ of $\mathcal{D}(G)$ such that supp $\psi_1 \subset V$ and $\mathcal{K} = \psi_1 * \mathcal{K}_2$.

This result is proven by adapting the preceding reasoning starting from lemma 4.2: in this lemma, instead of considering an infinitely differentiable map with compact support from U to $\mathcal{S}(\mathbf{Z})$, we consider a compact subset of $\mathcal{D}(U, \mathcal{S}(\mathbf{Z}))$; similarly modify lemmas 4.3 and 4.4; the details are left to the reader.

4.11. Theorem: *Let G, V, E, π, E_∞ and ξ be as in 3.3. Suppose that G satisfies the same condition as in 4.9. Then there exist $\psi \in \mathcal{D}(G)$ and $\eta \in E_\infty$ such that* $\operatorname{supp} \psi \subset V$ *and* $\xi = \pi(\psi)\eta$.

There exist $\varphi_1, \ldots, \varphi_n \in \mathcal{D}(G)$ and $\eta_1, \ldots, \eta_n \in E_\infty$ such that

$$\xi = \pi(\varphi_1)\eta_1 + \pi(\varphi_n)\eta_n$$

(th. 3.3). By the remark 4.10, there exist $\psi, \psi_1, \ldots, \psi_n \in \mathcal{D}(G)$ such that supp $\psi \subset V$ and $\varphi_1 = \psi * \psi_1, \ldots, \varphi_n = \psi * \psi_n$. Then

$$\xi = \pi(\psi) * (\pi(\psi_1)\eta_1 + \ldots + \pi(\psi_n)\eta_n) \ ,$$

which proves the theorem.

5. Some lemmas

The main goal of this chapter is to prove the lemmas 5.3, 5.4, 5.5, which will be useful in chapter 6.

5.1. Lemma: *Let $P \in \mathbf{C}[X, Y]$ be a polynomial. Let Γ be the curve in $\mathbf{C}^2$ whose equation is $P(\zeta_1, \zeta_2) = 0$. Suppose that*

(1) if Δ_1 is the line with equation $\zeta_1 = 0$ in $\mathbf{C}^2$, there exists $a_1 \in \Gamma \cap \Delta_1$ such that Γ is not tangent to Δ_1 at a_1;

(2) if Δ_2 is the line with equation $\zeta_2 = 0$ in $\mathbf{C}^2$, there exists $a_2 \in \Gamma \cap \Delta_1$, $a_2 \neq a_1$ such that Γ is not tangent to Δ_2 at a_2;

(3) P is irreducible and Γ is non-singular.

Let V_P be the set of $(z_1, z_2) \in \mathbf{C}^2$ such that $P(e^{z_1}, e^{z_2}) = 0$. Then V_P is a (non-singular) complex analytic subvariety of $\mathbf{C}^2$, and is convex.

Let θ be the map $(z_1, z_2) \mapsto (e^{z_1}, e^{z_2})$ from $\mathbf{C}^2$ to $\mathbf{C}^2$; this map is of rank 2 at every point, and defines $\mathbf{C}^2$ as a covering of $\mathbf{C}^2 - (\Delta_1 \cup \Delta_2)$. Since Γ is non-singular, V_P is a non-singular complex analytic subvariety of $\mathbf{C}^2$, and $\theta|_{V_P}$ defines V_P as a covering of $\Gamma - (\Delta_1 \cup \Delta_2)$. Therefore $\Gamma - (\Delta_1 \cup \Delta_2)$ is connected. The lemma will be proven by showing that two arbitrary coverings of V_P can be joined in a continuous way.

Let $z = (z_1, z_2) \in V_P$ and $\zeta = \theta(z)$. Consider a path $t \mapsto (\zeta_1(t), \zeta_2(t))$ in $\Gamma - (\Delta_1 \cup \Delta_2)$ which starts at ζ, goes to a point near a_1, turns around a_1, and comes back to ζ in the opposite direction; one can arrange so that the argument of $\zeta_1(t)$ is increased by $2\pi q (q \in \mathbf{Z})$ and that the argument of $\zeta_2(t)$ takes the same value. This path lifts uniquely to a continuous path γ in V_p starting from z and ending at $(z_1 + 2i\pi q, z_2)$. Reasoning in the same way for a_2, one obtains the result wanted.

5.2. Lemma: *We use the notation V_P of 5.1. Let $\mathcal{P}_n$ be the set of elements of $\mathbf{C}[X, Y]$ of degree $\leq n$. Then there exists an open dense subset $\mathcal{O}_n$ of $\mathcal{P}_n$ such that, for all $P \in \mathcal{O}_n$, the conditions of 5.1 are satisfied.*

This is well-known.

5.3. We denote by $\mathcal{M}$ the set of measures of $\mathbf{R}^2$ satisfying the following properties:

(a) the support of μ is a finite subset of $\mathbf{Q}^2$; it follows then that the Fourier transform $\hat{\mu}(\zeta_1, \zeta_2)$ of μ is of the form

$$e^{2i\pi(\alpha_1\zeta_1 + \alpha_2\zeta_2)} P(e^{-2i\pi\alpha\zeta_1}, e^{-2i\pi\alpha\zeta_2}) ,$$

where $\alpha_1, \alpha_2, \alpha \in \mathbf{Q}$ and $P \in \mathbf{C}[X, Y]$;

(b) the polynomial P satisfies the conditions listed in 5.1; it follows then that $\hat{\mu}^{-1}(0)$ is a connected non-singular complex analytic subvariety of $\mathbf{C}^2$.

Let T denote the triangle in $\mathbf{R}^2$ whose corners are the points

$$\left(\frac{2}{3},-\frac{1}{3}\right),\left(-\frac{1}{3},\frac{2}{3}\right),\left(-\frac{1}{3},-\frac{1}{3}\right) .$$

Let j be the positively homogeneous gauge function on $\mathbf{R}^2$ such that $T=\{x : j(x)\leq 1\}$. We denote by $B(0,r)$ the ball rT centered at 0 and of radius r associated to this gauge.

Lemma: *Let A be a finite subset of $\mathbf{Q}^2\cap B(x_0,r)$, ν be a measure on $\mathbf{R}^2$ whose support is contained in A, and $\varepsilon>0$. Then there exists a measure $\mu\in\mathcal{M}$ such that $\|\nu-\mu\|\leq\varepsilon$ and $A\subset\operatorname{supp}(\mu)\subset B(x_0,r)$.*

By homothety and translation, one can suppose that $A\subset\mathbf{N}^2$.

Let $\nu=\sum_{(k,l)\in A}\alpha_{kl}\delta_{(k,l)}$. We will search for μ to be of the form

$$\sum_{(k,l)\in A}\beta_{kl}\delta_{(k,l)} ,$$

where β_{kl} are non-zero complex numbers. One has

$$\hat{\mu}(\zeta_1,\zeta_2)=\sum_{k+l<r'}\beta_{kl}e^{-2i\pi k\zeta_1}e^{-2i\pi l\zeta_2} .$$

It is necessary that $\sum_{(k,l)\in A}|\beta_{kl}-\alpha_{kl}|<\varepsilon$ and that the polynomial $\sum_{k+l<r'}\beta_{kl}X^kY^l$ satisfies the conditions of 5.1. This is possible by 5.2.

5.4. We choose a function $h\in\mathcal{D}(\mathbf{R}^2)$ such that $\int_{\mathbf{R}^2}h=1$. For any $\eta>0$, let $h_\eta(\xi)=\eta^{-2}h(\xi\eta^{-1})$ so that $h_\eta\in\mathcal{D}(\mathbf{R}^2)$; h_η's form an approximate identity.

Lemma: *Let $\psi\in\mathcal{D}^k(\mathbf{R}^2), x_0\in\mathbf{R}^2, r>0$ such that $\operatorname{supp}\psi\subset B(x_0,r)$. Let $\eta_0>0,\varepsilon>0$. Then there exist $\eta\in(0,\eta_0)$ and $\mu\in\mathcal{M}$ (see 5.3) such that*

$$\|h_\eta*\mu-\psi\|_k<\varepsilon,$$

$$\operatorname{co}(\psi)\subset\operatorname{co}(\mu)\subset B(x_0,r+\eta).$$

The approximate identity defines a convolution operator on $\mathcal{D}^k(\mathbf{R}^2)$ which converges strongly to the identity. Hence there exists $\eta\in(0,\eta_0)$ such that

$$\|h_\eta*\psi-\psi\|_k<\frac{\varepsilon}{3}.$$

In addition, let v_λ be the discretization of ψ:

$$v_\lambda=\sum_{(n_1,n_2)\in\mathbf{Z}^2}\delta_{(n_1\lambda,n_2\lambda)}\int_{n_1\lambda}^{(n_1+1)\lambda}\int_{n_2\lambda}^{(n_2+1)\lambda}\psi,$$

where $\lambda > 0, \lambda \in \mathbf{Q}$. One then has $\mathrm{co}(v_\lambda) \subset \mathrm{co}(\psi) + B(0, 3\lambda)$. The evaluation of mean values(?) gives

$$\|h_\eta * \psi - h_\eta * v_\lambda\|_k \leq \lambda \|h_\eta\|_{k+1} \int |\psi|.$$

Fix $\lambda < \eta/3$ such that the second term is $< \varepsilon/3$. Let A be a finite subset of $\mathbf{Q}^2$ whose convex hull contains $\mathrm{co}(\psi)$ and $\mathrm{co}(v_\lambda)$, and is contained in $B(x_0, r+\eta)$. There exists $\mu \in \mathcal{M}$ supported on A such that

$$\|v_\lambda - \mu\| < \frac{\varepsilon}{3}(\|h_\eta\|_k)^{-1}$$

(lemma 5.3). Then μ possesses all of the properties listed in the lemma.

5.5. If $r > 0$, let (see chap. 1 for the notation)

$$\Gamma_r = \{f \in \mathcal{D}^0(\mathbf{R}^2) | \, \exists x_0 \text{ such that } \mathrm{co}(f) \supset B(x_0, r)\},$$

$$\Delta_r = \{f \in \mathcal{D}^0(\mathbf{R}^2) | \, \exists y_0 \text{ such that } \mathrm{co}(f) \subset B(y_0, r)\}.$$

Recall that, if $f_1, f_2 \in \mathcal{D}^0(\mathbf{R}^2)$, one has

$$\mathrm{co}(f_1 * f_2) = \mathrm{co}(f_1) + \mathrm{co}(f_2). \tag{14}$$

Lemma: *Let $f_1, f_2 \in \mathcal{D}^0(\mathbf{R}^2)$. Let r, r' be such that $0 < r' < r$. Then*
(i) if $f_1 * f_2 \in \Gamma_r$ and $f_1 \in \Delta_{r'}$, one has $f_2 \in \Gamma_{r-r'}$;
(ii) if $f_1 * f_2 \in \Delta_r$ and $f_1 \in \Gamma_{r'}$, one has $f_2 \in \Delta_{r-r'}$.

(i) By translation, one can suppose that $\mathrm{co}(f_1 * f_2) \supset B(0, r)$ and $\mathrm{co}(f_1) \subset B(0, r')$. By (14), one then has

$$B(0, r) \subset B(0, r') + \mathrm{co}(f_2).$$

Suppose that $\mathrm{co}(f_2) \not\supset B(0, r - r')$. Then $\mathrm{co}(f_2)$ has a support line intersecting the triangle $(r - r')T$. If

$$L = \{l \in (\mathbf{R}^2)^*; \max_{\xi \in T} l(\xi) = 1\},$$

one can find $l \in L$ and $\varepsilon > 0$ such that

$$\mathrm{co}(f_2) \subset \{\xi; l(\xi) < r - r' - \varepsilon\}.$$

Using the identity

$$\sup(l(A + B)) = \sup(l(A)) + \sup(l(B)),$$

one deduces

$$\sup l(B(0, r') + \mathrm{co}(f_2)) \leq r' + r - r' - \varepsilon = r - \varepsilon,$$

which is a contradiction.

(ii) The proof reduces to the case where

$$B(0,r) \supset B(0,r') + \mathrm{co}(f_2).$$

Suppose $\mathrm{co}(f_2) \not\subset B(0,r-r')$; then there exist $l \in L$ and $\varepsilon > 0$ such that

$$\sup l(\mathrm{co}(f_2)) = r - r' + \varepsilon;$$

$$\sup l(B(0,r') + \mathrm{co}(f_2)) = r + r - r' + \varepsilon = r + \varepsilon.$$

6. Groups with strong factorization

6.1. Theorem: *There exists a function in $\mathcal{D}(\mathbf{R}^2)$ which is not the convolution product of two functions in $\mathcal{D}(\mathbf{R}^2)$.*

(a) The theorem 6.1 is proven using a contradiction: suppose that $\mathcal{D}(\mathbf{R}^2) * \mathcal{D}(\mathbf{R}^2) = \mathcal{D}(\mathbf{R}^2)$. By (14), we then have

$$\mathcal{D}_1(\mathbf{R}^2) * \mathcal{D}_1(\mathbf{R}^2) \supset \mathcal{D}_1(\mathbf{R}^2) \tag{15}$$

(we denote by $\mathcal{D}_1(\mathbf{R}^2)$ the set of $\varphi \in \mathcal{D}(\mathbf{R}^2)$ such that $\mathrm{supp}\,\varphi \subset B(0,1)$; the notation $\mathcal{D}_1^k(\mathbf{R}^2)$ is defined similarly). For any integer $n > 0$, with the notation of 5.5, let

$$\begin{aligned} F_n &= \{\varphi \in \mathcal{D}_1(\mathbf{R}^2) \cap \Gamma_{1/n} \mid \|\varphi\|_1 \leq n\}, \\ F'_n &= \{\varphi \in \mathcal{D}_1^0(\mathbf{R}^2) \cap \Delta_{1-(1/n)} \mid \|\varphi\|_0 \leq n\}. \end{aligned}$$

(b) We establish the following result:

There exist k, $n_0 \in \mathbf{N}$, $\varepsilon > 0$, and $\varphi_0 \in \mathcal{D}_1(\mathbf{R}^2)$ such that

$$\Omega = \{\varphi \in \mathcal{D}_1^k(\mathbf{R}^2) | \|\varphi - \varphi_0\|_k < \varepsilon\} \subset F'_{n_0} * F'_{n_0}.$$

By (15), one has $\mathcal{D}_1(\mathbf{R}^2) \subset \cup_{n \geq 1} F_n * F_n$. Let

$$B_n = (F_n * F_n) \cap \mathcal{D}_1(\mathbf{R}^2).$$

By the Baire theorem, there exists n_0 such that $\mathrm{adh}_{\mathcal{D}_1(\mathbf{R}^2)}(B_{n_0})$ contains a non-empty open subset of $\mathcal{D}_1(\mathbf{R}^2)$. Hence there exist $k \in \mathbf{N}, \varepsilon > 0$ and $\varphi_0 \in \mathcal{D}(\mathbf{R}^2)$ such that

$$\mathrm{adh}_{\mathcal{D}_1(\mathbf{R}^2)}(B_{n_0}) \supset \{ \|\varphi \in \mathcal{D}_1(\mathbf{R}^2) \mid \|\varphi - \varphi_0\|_k < \varepsilon\},$$

and

$$\mathrm{adh}_{\mathcal{D}_1^k(\mathbf{R}^2)}(B_{n_0}) \supset \{ \|\varphi \in \mathcal{D}_1^k(\mathbf{R}^2) \mid \|\varphi - \varphi_0\|_k < \varepsilon\} = \Omega.$$

Let $\psi \in \Omega$. Then ψ is the limit in $\mathcal{D}_1^k(\mathbf{R}^2)$ of a sequence (φ_p) where $\varphi_p \in B_{n_0}$ for all p. One has $\varphi_p = u_p * v_p$ where $u_p, v_p \in F_{n_0}$. By Ascoli's Theorem, we can replace the sequences (u_p)

and (v_p) with uniformly convergent subsequences. Let u, v (resp.) be the limits of $(u_p), (v_p)$ (resp.) in $\mathcal{D}_1^0(\mathbf{R}^2)$. Then φ_p converges uniformly to $u * v$, where $\psi = u * v$. By 5.5 (ii), one has $u_p \in \Delta_{1-(1/n_0)}, v_p \in \Delta_{1-(1/n_0)}$. Since

$$\operatorname{supp}(u) \subset \liminf(\operatorname{supp}(u_p)),$$

one deduces that $u \in \Delta_{1-(1/n_0)}$. Similarly, $v \in \Delta_{1-(1/n_0)}$, so that $\psi \in F'_{n_0} * F'_{n_0}$.

(c) There exist $\rho \in (0, 1/n_0)$ and $\psi \in \Omega$ such that $\operatorname{co}(\psi) = B(0, 1-\rho)$.

Using 5.4 and its notation, one can find $\eta \in (0, \rho)$ and $\mu \in \mathcal{M}$ such that

$$h_\eta * \mu \in \Omega, \qquad \operatorname{co}(\mu) \supset B(0, 1-\rho).$$

Since $\Omega \subset F'_{n_0} * F'_{n_0}$, there exist $u, v \in F'_{n_0}$ such that $h_\eta * \mu = u * v$.

Then $\hat{u}\hat{v}$ vanishes on the connected complex analytic variety $\hat{\mu}^{-1}(0)$; by interchanging u and v if needed, we can suppose that

$$\hat{u}^{-1}(0) \supset \hat{\mu}^{-1}(0).$$

(d) Define L as in 5.5. Let $g(l) = \inf_{\xi \in T} l(\xi)$.

If $l \in L$, denote by μ^l the image of the measure μ on $\mathbf{R}$ under the map $x \mapsto l(x)$; similarly, denote by u^l the image of the measure $u(x)dx$ under the same map.

If θ is a non-zero measure on $\mathbf{R}$ with compact support, and if $r > 0$, let

$N_\theta(r) =$ the number of zeros of $\hat{\theta}(\zeta)$, counted with multiplicity, in the disk $\{\zeta \in \mathbf{C} \mid |\zeta| < r\}$;

$$N^*_\theta(r) = \text{ cardinality of the set } \hat{\theta}^{-1}(0) \cap \{\zeta \in \mathbf{C} \mid |\zeta| < r\}.$$

By a classical result ([2], p. 114-116 and [11], p.13), one has

$$\lim_{r\to\infty} \frac{1}{r} N_\theta(r) = \text{ length of } \operatorname{co}(\theta). \tag{16}$$

(e) Let L_1 be the set of $l \in L$ such that the support lines of $\operatorname{co}(\mu)$ associated to $\pm l$ intersect $\operatorname{co}(\mu)$ at only one point. Let L_2 be the set of $l = (\alpha_1, \alpha_2) \in L$ such that $\alpha_2 \neq 0, \alpha_1/\alpha_2 \notin \mathbf{Q}$.

We now establish the following results:

(i) if $l \in L_1$, one has $\operatorname{co}(\mu^l) \supset (1-\rho)[g(l), 1]$;

(ii) if $l \in L_2$, one has $N^*_{\mu^l}(r) = N_{\mu^l}(r) + O(1)$ as $r \to \infty$.

The assertion (i) results from the fact that $\operatorname{co}(\mu) \supset B(0, 1-\rho)$ and by the definition of L_1.

Let $l = (\alpha_1, \alpha_2) \in L_2$. The system

$$(\mu^l)\hat{}(\zeta) = 0, \qquad \frac{d}{d\zeta}(\mu^l)\hat{}(\zeta) = 0,$$

can be written as

$$\begin{cases} P(\zeta_1,\zeta_2)=0, \qquad (\alpha_1\zeta_1 P'_{\zeta_1}+\alpha_2\zeta_2 P'_{\zeta_2})(\zeta_1,\zeta_2)=0, \\ \qquad\qquad \zeta_1 = e^{i\alpha_1\zeta}, \qquad \zeta_2 = e^{i\alpha_2\zeta}, \end{cases}$$

where P is an irreducible polynomial (see the definition of $\mathcal{M}$). The first two equations are not satisfied by a finite number of points (ζ_1,ζ_2). Since the map

$$\zeta \mapsto (e^{i\alpha_1\zeta}, e^{i\alpha_2\zeta})$$

from $\mathbf{C}$ to $\mathbf{C}^2$ is injective, (ii) is established.

(f) Let $l \in L_1 \cap L_2$. Suppose $u^l \neq 0$. If $k(l) = 1 - g(l)$, one has

$$\begin{aligned} (1-\rho)k(l) &\leq \lim_{r\to\infty} \frac{1}{r} N_{\mu^l}(r), \text{ by (16) and (e), (i)} \\ &= \lim_{r\to\infty} \frac{1}{r} N^*_{\mu^l}(r), \text{ by (e), (ii)} \\ &\leq \limsup_{r\to\infty} \frac{1}{r} N^*_{u^l}(r) \text{ by (c)} \\ &\leq \lim_{r\to\infty} \frac{1}{r} N_{u^l}(r). \end{aligned}$$

Now, $u \in \Delta_{1-(1/n_0)}$, hence $\mathrm{co}(u^l) \subset [1-(1/n_0)][g(l),1]$ and therefore, by (16)

$$\lim_{r\to\infty} \frac{1}{r} N_{u^l}(r) \leq \Big(1 - \frac{1}{n_0}\Big) k(l).$$

We thus obtain a contradiction when $\rho < 1/n_0$. Hence $u^l = 0$ for all $l \in L_1 \cap L_2$. By continuity, $u^l = 0$ for all l, and hence $\hat{u} = 0$, $u = 0$, and $h_\eta * \mu = 0$. This is absurd when $\mathrm{co}(\mu) \supset B(0, 1-\rho)$.

6.2. Lemma: *Let G be a Lie group and H a closed distinguished subgroup of G. Suppose that $\mathcal{D}(G) = \mathcal{D}(G) * \mathcal{D}(G)$. Then*

$$\mathcal{D}(G/H) = \mathcal{D}(G/H) * \mathcal{D}(G/H).$$

Let $\pi : G \to G/H$ be the canonical map. For all $\varphi \in \mathcal{D}(G)$, let $A\varphi$ be the element of $\mathcal{D}(G/H)$ defined by

$$(A\varphi)(\pi x) = \int_H \varphi(xy)dy$$

for all $x \in G$ (dy denotes a left Haar measure on H). Then for ai suitable choice of Haar measures on G and G/H, A is a homomorphism of $\mathcal{D}(G)$ onto $\mathcal{D}(G/H)$, and hence the lemma.

6.3. Let G be a Lie group. It results from 6.1 and 6.2 that, if G admits a quotient group isomorphic to $\mathbf{R}^2$, then $\mathcal{D}(G) \neq \mathcal{D}(G) * \mathcal{D}(G)$. This is the case when G is simply connected nilpotent of dimension ≥ 2.

7. The case of simply connected nilpotent groups

7.1. Theorem: *Let G be a simply connected nilpotent Lie group and $\varphi \in \mathcal{D}(G)$. Then there exist $\chi \in \mathcal{D}(G)$ and $\psi \in \mathcal{S}(G)$ such that $\varphi = \psi * \chi$ and supp $(\chi) \subset \mathrm{supp}(\varphi)$.*

Let $\mathfrak{g}$ be the Lie algebra of G. Let $(\mathfrak{g}_0, \mathfrak{g}_1, \ldots, \mathfrak{g}_{\mathfrak{m}})$ be an increasing sequence of ideals of $\mathfrak{g}$ of dimensions $0, 1, \ldots, m = \dim \mathfrak{g}$. Let $x_i \in \mathfrak{g}_i$ be such that $x_i \notin \mathfrak{g}_{i+1}$. The map

$$\zeta : (t_1, \ldots, t_m) \mapsto (\exp t_1 x_1) \cdots (\exp t_m x_m))$$

from $\mathbf{R}^m$ to G is then a diffeomorphism from $\mathbf{R}^m$ onto G; moreover, ζ transforms $\mathcal{S}(\mathbf{R}^m)$ to $\mathcal{S}(G)$ and the Lebesgue measure on $\mathbf{R}^m$ to the measure $P \cdot \beta$, where β is a Haar measure on G and P is a polynomial on G.

Reasoning as in theorem 3.1, one constructs a function $f \in \mathcal{S}(\mathbf{R})$ and positive numbers $\alpha_0, \alpha_1, \alpha_2, \ldots$ such that, denoting the image of the measure $f(t_1)dt_1$ by μ, one has

$$\mu * \sum_{n=0}^{p} (-1)^n \alpha_n x_1^{2n} * \varphi \to \varphi \quad \text{in } \mathcal{S}'(G),$$

$$\sum_{n=0}^{p} (-1)^n \alpha_n x_1^{2n} * \varphi \to \theta \quad \text{in } \mathcal{D}(G).$$

Then $\varphi = \mu * \theta$ and supp $(\theta) \subset \mathrm{supp}(\varphi)$.

Continuing this way, one obtains $\varphi = \xi_1 * \cdots * \xi_m * \chi$, where

$$\chi \in \mathcal{D}(G)\,,\ \mathrm{supp}(\chi) \subset \mathrm{supp}(\varphi)$$

and where ξ_i is the image under the map $t_i \mapsto \exp t_i x_i$ of a measure of the form $f_i(t_i)dt_i$, with $f_i \in \mathcal{S}(\mathbf{R})$. The function $(t_1, \ldots, t_m) \mapsto f_1(t_1) \cdots f_m(t_m)$ on $\mathbf{R}^m$ belongs to $\mathcal{S}(\mathbf{R}^m)$, hence $\xi_1 * \cdots * \xi_m$ is of the form $\xi P \beta$, where $\xi \in \mathcal{S}(G)$. However, $\xi P \in \mathcal{S}(G)$, and this proves the theorem.

7.2. Theorem: *Let G be a simply connected nilpotent Lie group, V a neighborhood of e in G, and $\varphi \in \mathcal{S}(G)$. Then*

(i) φ is a finite sum of functions of the form $\psi_1 * \psi_2$, where $\psi_1 \in \mathcal{D}(G)$, $\psi_2 \in \mathcal{S}(G)$, $\mathrm{supp}(\psi_1) \subset V$, $\mathrm{supp}(\psi_2) \subset \mathrm{supp}(\varphi)$;

(ii) φ is of the form $\chi_1 * \chi_2$ where $\chi_1, \chi_2 \in \mathcal{S}(G), \mathrm{supp}(\chi_2) \subset \mathrm{supp}(\varphi)$.

The proof proceeds analogous to the proofs of 3.1 and 7.1.

7.3. Corollary: *Let π be an irreducible continuous unitary representation of G, $\varphi \in \mathcal{S}(G)$ and (λ_n) the decreasing sequence of the eigenvalues of $\pi(\varphi) * \pi(\varphi)$ (counted with multiplicity). Then the sequence (λ_n) is of rapid decay.*

It is known that $\pi(\varphi)$ is of trace-class. The rest of the proof proceeds as in 3.2.

7.4. Theorem: *Let G be a simply connected nilpotent Lie group, E a Hilbert space, π a continuous unitary representation of G on E, E_∞ the set of smooth vectors in E for π, and $\xi \in E_\infty$. Then there exist $\eta \in E_\infty$ and $\psi \in \mathcal{S}(G)$ such that $\xi = \pi(\psi)\eta$.*

Adopting the proofs of 3.3 and 7.1 yields the theorem.

(This result is mentioned briefly in [9] when π is irreducible. The general case does not seem to simply reduce to the irreducible case.)

7.5. Corollary: *Let G, E, π, E_∞ be as in 7.4, and $v \in \mathcal{O}'_c(G)$. Then there exists a unique linear map $A : E_\infty \to E_\infty$ such that*

$$A(\pi(\psi)\eta) = \pi(v * \psi)\eta,$$

for every $\psi \in \mathcal{S}(G)$ and $\eta \in E$. The map A is continuous when E_∞ is equipped with the Fréchet topology.

The uniqueness of A results at once from theorem 7.4.

Let (v_n) be a sequence of elements in $\mathcal{E}'(G)$ converging to v in $\mathcal{O}'_c(G)$. Recall (see for example [4], p. 24) that $\pi(v_n) : E_\infty \to E_\infty$ are defined and continuous. We show that $\pi(v_n)$ converges pointwise to a limit. Any element of E_∞ can be written as $\pi(\psi)\eta$ where $\psi \in \mathcal{S}(G)$ and $\eta \in E$ (th. 7.4). For any $u \in U(\mathfrak{g})$, the vector

$$\pi(u)\pi(v_n)\pi(\psi)\eta = \pi(u * v_n * \psi)\eta$$

converges in E to $\pi(u * v * \psi)\eta$ (note that $u * v * \psi \in \mathcal{S}(G)$). Hence $\pi(v_n)\pi(\psi)\eta$ converges in E_∞ to $\pi(v * \psi)\eta$.

By Banach-Steinhaus theorem, there exists a continuous linear map $A : E_\infty \to E_\infty$ such that $\pi(v_n)$ converges pointwise to A; with the previous notation, one has

$$A\pi(\psi)\eta = \pi(v * \psi)\eta.$$

7.6. We continue using the notation in 7.5. It is natural to denote the endomorphism A by $\pi(v)$. One then has

$$\pi(v)\pi(\psi) = \pi(v * \psi)$$

for any $v \in \mathcal{O}'_c(G)$ and any $\psi \in \mathcal{S}(G)$. This definition of $\pi(v)$ extends the current definition for $v \in \mathcal{E}'(G)$ and $v \in \mathcal{S}(G)$.

One can show that $\mathcal{O}'_c(G)$ is an algebra under the convolution, and that $v \mapsto \pi(v)$ is a homomorphism of algebras.

7.7. We still use the notation in 7.5. Recall that, for

$$v \in \mathcal{E}'(G), \xi \in E_\infty,$$

one has

$$(\pi(v)\xi \,|\, \zeta) = \int_G (\pi(s)\xi \,|\, \zeta) dv(s) \tag{17}$$

the integral being defined when the function $s \mapsto (\pi(s)\xi \,|\, \zeta)$ belongs to $\mathcal{E}(G)$.

It would have been natural to define to also define $\pi(v)$ for $v \in \mathcal{O}'_c(G)$ by the equation (16). However, $\mathcal{O}'_c(G)$ does not have a canonical duality with the space $\mathcal{O}_c(G)$ of infinitely differentiable, very slowly decaying functions ([8], loc. cit.). Now, the function $s \mapsto (\pi(s)\xi \,|\, \zeta)$ on G (while being slow decaying) is not in general very slowly decaying, as one can see via an example. The fact that one can nevertheless define $\pi(v)$ means that we have a summation procedure for the integral (17).

Take G to be the 3-dimensional Heisenberg group. We identify it with its Lie algebra by the exponential map; and by $\mathbf{R}^3$; the product in G is defined by

$$(x, y, z)(x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - yx')\right).$$

There exists an irreducible unitary representation π of G in $L^2(\mathbf{R})$ defined by

$$(\pi(x, y, z)f)(\theta) = e^{i(z+y\theta+(1/2)xy)} f(\theta + x)$$

for $x, y, z, \theta \in \mathbf{R}$ and $f \in L^2(\mathbf{R})$. The smooth vectors for π are the elements of $\mathcal{S}(\mathbf{R})$. Let $f \in \mathcal{S}(\mathbf{R})$ be such that $f(\theta) = 1$ on $[-1, 2]$; let $g \in L^2(\mathbf{R})$ be the characteristic function of $[0, 1]$. Then

$$(\pi(x, y, z)f \,|\, g) = \int_{\mathbf{R}} e^{i(z+y\theta+(1/2)xy)} f(\theta + x)\overline{g(\theta)} d\theta.$$

Let $\alpha(x,y,z)$ be this integral. If $x \in [-1,1]$, one has

$$\begin{aligned}\alpha(x,y,z) &= \int_0^1 e^{i(z+y\theta+(1/2)xy)}d\theta \\ &= e^{i(z+(1/2)xy)}\frac{e^{iy}-1}{iy},\end{aligned}$$

hence

$$\frac{\partial^n \alpha(0,y,z)}{\partial x^n} = \left(\frac{1}{2}iy\right)^n e^{iz}\frac{e^{iy}-1}{iy} = 2^{-n}(iy)^{n-1}e^{iz}(e^{iy}-1).$$

Therefore, for every $k \geq 0$, there exists an n such that the function

$$(1+x^2+y^2+z^2)^{-k}\,\frac{\partial^n \alpha(x,y,z)}{\partial x^n}$$

does not approach 0 at infinity. This proves that α does not decay very slowly.

Appendix

We now explain how the results in section 2 can be extended to functions invariant on balls on $\mathbf{R}^n$.

For $x \in \mathbf{R}^n$, let $r = ((x_1)^2 + \cdots + (x_n)^2)^{1/2}$. Using the notation of 2, we let $\tilde{\chi}_\lambda(x) = \chi_\lambda(r)$. Since χ_λ is an even function, it follows that $\tilde{\chi}_\lambda$ is the restriction to $\mathbf{R}^n$ of a meromorphic function on $\mathbf{C}^n$.

On the other hand, by 2.3, $\chi_\lambda \in \mathcal{S}(\mathbf{R})$ and using the theorem on composite functions, $\tilde{\chi}_\lambda \in \mathcal{S}(R^n)$. We denote the Fourier transform of $\tilde{\chi}_\lambda$ by ψ_λ.

Let $\mathfrak{g}_2$ be the Gevrey class and d_K be the distance function for the restrictions of functions in $\mathfrak{g}_2$ to K:

$$d_K(0,f) = \sup_{x\in K, m\in \mathbf{N}} \left[\left|(m!)^{-2}\frac{\partial^\alpha}{\partial x^\alpha}f(x)\right|\right]^{1/m} \text{ where } |\alpha| \leq m.$$

Given a compact set K not containing the origin, $d_K(0,\psi_\lambda)$ is bounded independent of λ.

Using a partition of unity, it suffices to bound $d_K(0,u\psi_\lambda)$ where $u \in \mathfrak{g}_2$, *fixed*, support$(u) \subset \prod_k\{x_k > 0\}$; then the Fourier transform v of u satisfies that

$$|v(x_1 - i\eta, x_2, \ldots, x_n)| < c_1 \exp(-\varepsilon\eta - c_2\|x\|^{1/2}) \qquad (\varepsilon, c_2 > 0).$$

The bound at infinity of $v * \tilde{\chi}_\lambda$ depends only on the bound of

$$h(x) = \int_{\|x-z\|<1/2\|x\|} v(x-z)\tilde{\chi}_\lambda(z)dz \qquad (\text{ where } \|x\| = \sup|x_k|).$$

Suppose $x_1 = \|x\|$, and let $z = (z_1, \tilde{z})$, $z_1 \in \mathbf{R}$, $\tilde{z} \in \mathbf{R}^{n-1}$ and integrate with respect to z_1 ; we obtain

$$h_1(x,z) = \int_{1/2x_1}^{3/2x_1} v_{\tilde{x}-\tilde{z}}(x_1 - z_1)p_\lambda(\|\tilde{z}\|^2 + z_1^2)dz_1$$

where $p_\lambda(\alpha) = \chi_\lambda(\alpha^{1/2})$.

Complexify the variable $z_1 : z_1 \to \zeta = \xi + i\eta$ and let

$$\Gamma = \left\{\zeta \in \mathbf{C}; \eta > 0, \xi \in \left[\frac{1}{2}x_1, \frac{3}{2}x_1\right]\right\};$$

the function $p_\lambda(\|\tilde{z}\|^2 + \zeta^2)$ is holomorphic in Γ, and one can write h_1 as an integral along the two vertical sides of Γ, where the bound $|h_1(x,z)| < c_3 \exp(-c_4 \|\!|x|\!\|^{1/2})$ is conserved under integration with respect to $\tilde{z}$.

It then results by theorem 3.1 that, if $G = \mathbf{R}^n$, the 'finite sum' can be reduced to a sum of *two* terms (this improves [12]).

References

[1] Bernat (P.), Conze (N.), ... – *Représentations des groupes de Lie résolubles*, Paris, Dunod, 1972.

[2] Boas (P.). – *Entire functions*, Academic Press, 1954.

[3] Bourbaki (N.). – *Variétés différentielles et analytiques.* – Paris, Hermann, 1971 (Act. scient. et ind., 1333).

[4] Cartier (P.). – Vecteurs différentiables dans les représentations unitaires des groupes de Lie, *Lecture Notes in Math.*, No. 514, 1974/1975, p. 20-34.

[5] Duflo (M.) and Labesse (J. P.) – Sur la formule des traces de Selberg, *Ann. Scient. Éc. Norm. Sup.*, Vol. 4, 1971, p. 193-284.

[6] Dunford (N.) and Schwartz (J. T.) – *Linear operators*, part II, Interscience publishers, 1963.

[7] Ehrenpreis (L.). – Solution of some problems of division IV, *Amer. J. Math.*, Vol. 82, 1960, p. 552-588.

[8] Grothendieck (A.). – Produits tensoriels topologiques et spaces nucléaires, *Memoirs of the Amer. Math. Soc.*, No. 16, 1955.

[9] Howe (R.). – *On a connection between nilpotent groups and oscillatory integrals associated to singularities*, preprint.

[10] Jenkins (J. W.). – *Primary projections on L^2 of a nilmanifold*, preprint.

[11] Levinson (N.) – Gap and density theorems, *Amer. Math. Soc. Coll. Pub.*, Vol. 26, 1940.

[12] Rubel (L. A.), Squires (W. A.) and Taylor (B. A.). – *Irreducibility of certain entire functions with applications to harmonic analysis*, to appear in *Annals of Math.* (M. Lee Rubel announced these results on May 3, 1977 at Séminaire d'Analyse de P. Lelong.)