RESIDUES OF EISENSTEIN SERIES VIA MAASS-SELBERG RELATIONS

FERYÂL ALAYONT

Abstract. Using the Maaß-Selberg relations, it is shown that Eisenstein series induced from cuspidal data on a maximal parabolic subgroup of a reductive group has poles only when the data is self-associate and the poles in the positive Weyl chamber lie on the real line inside the non-convergent region. In the rank-2 parabolic subgroup case for $G = GL_n$ it is shown that only the maximal self-associate data can give rise to square-integrable residues. The information for this case implies that the maximal parabolic subgroup Eisenstein series has a single pole in the region in the positive Weyl chamber.

Introduction

Let G be a quasi-split reductive group defined over a number field k and $\mathbf{A}$ the adele ring of k. The spectral decomposition of the right regular representation on $L^2(G_k\backslash G_{\mathbf{A}})$ [Langlands 1976] consists of the direct sum of irreducible cuspidal representations and the direct integrals of Eisenstein series induced from cuspidal data on parabolic subgroups of G and their residues. The residual spectrum, the part consisting of the square-integrable residues of Eisenstein series, for $G = GL(n)$ is completely analyzed in [Mœglin-Waldspurger 1989] proving Jacquet's conjecture [Jacquet 1983]. In this paper we use the Maaß-Selberg relations to determine when and where the maximal parabolic Eisenstein series for a general G can have poles and to reconstruct partial results from [Mœglin-Waldspurger 1989] concerning the Eisenstein series induced from next-to-maximal parabolic subgroups in $G = GL(n)$.

Let P be a parabolic subgroup of G and $P = MN$ be the Levi decomposition of P. Given a cuspform f on $M_k\backslash M_{\mathbf{A}}$ and $\lambda \in (\mathfrak{a}_P^G)^*_{\mathbf{C}}$, the Eisenstein series induced from this data on P is

$$E(g) = \sum_{\gamma \in P_k\backslash G_k} \varphi(\gamma g)$$

where the kernel function φ is defined by

$$\varphi(g) = \varphi(nmk) = \lambda(m) f(m) \delta_P^{1/2}(m)$$

using the Iwasawa decomposition $g = nmk$ of $g \in G_{\mathbf{A}}$. Applying the truncation operator $\wedge^T$ defined in [Arthur 1980] to the Eisenstein series, a square-integrable function is obtained. The Maaß-Selberg relations give explicit formulas for the inner products of these truncated Eisenstein series.

Suppose P is a maximal parabolic subgroup of G. The Maaß-Selberg relation in this case involves only two or four terms, depending on whether the data is self-associate or not. The poles of the Eisenstein series and those of the truncated Eisenstein series occur at the poles of the constant terms of the Eisenstein series. Comparing the residues on both sides of the Maaß-Selberg relation and using the fact that $\wedge^T E$ approaches E as $T \to \infty$, we show that the Eisenstein series does not have a pole unless the parabolic and the data are self-associate. In the self-associate case, the pole is on the real line and is of order 1.

We then consider a rank-2 parabolic subgroup P in $G = GL(n)$, i.e. P consists of block-upper-triangular matrices with three blocks on the diagonal. The Maaß-Selberg relations involve four or 36 terms. From the theory of the intertwining operators, the coefficients of the constant terms for this case are products of the coefficients in the maximal parabolic case. Again comparing the residues on both sides of the Maaß-Selberg relation and taking the limit $T \to \infty$, it is shown that the only square-integrable residues are the multi-residues in the case of a maximally self-associate data, i.e. the block sizes are all equal with the cuspform being the same on each block.

Acknowledgment: The author would like to thank Paul Garrett for his help and advice.

1. Notation and terminology

We follow partially the notation in [Arthur 1978] and [Arthur 1980]. Let G be a quasi-split reductive group over a number field k. Fix a minimal parabolic subgroup P_0 of G and a Levi subgroup M_0 of P_0, both over k. From now on, all parabolic (respectively Levi) subgroups will be *standard*, i.e. will contain P_0 (respectively M_0). Let M_P and N_P denote the (standard) Levi component and the unipotent radical of P. Let A_P be the split component of the center of M_P. Let $X(M_P)$ be the group of characters of M_P defined over k, and let

$$\mathfrak{a}_P^* = X(M_P) \otimes_{\mathbf{Z}} \mathbf{R} \qquad \text{and} \qquad \mathfrak{a}_P = \mathrm{Hom}_{\mathbf{Z}}(X(M_P), \mathbf{R}) \; .$$

Then $\mathfrak{a}_P^*$ and $\mathfrak{a}_P$ are naturally dual to each other and $\mathfrak{a}_P^* = X(A_P) \otimes_{\mathbf{Z}} \mathbf{R}$. Restricting a character of G to M_P gives an injection $\mathfrak{a}_G^* \hookrightarrow \mathfrak{a}_P^*$ and restricting a character on A_P to A_G gives a surjection splitting the exact sequence

$$0 \;\to\; \mathfrak{a}_G^* \;\to\; \mathfrak{a}_P^* \;\to\; \mathfrak{a}_P^*/\mathfrak{a}_G^* \;\to\; 0$$

We thus obtain the decompositions $\mathfrak{a}_P^* = (\mathfrak{a}_P^G)^* \oplus \mathfrak{a}_G^*$ and $\mathfrak{a}_P = \mathfrak{a}_P^G \oplus \mathfrak{a}_G$ dually.

Let Δ_0 be the simple roots of (P_0, A_0). Then Δ_0 is canonically embedded in $\mathfrak{a}_{P_0}^* = \mathfrak{a}_0^*$ and the parabolic subgroups P are in bijection with the subsets Δ_0^P of Δ_0 consisting of the roots which vanish on $\mathfrak{a}_P$. Let Δ_P be the restrictions to $\mathfrak{a}_P$ of elements in $\Delta_0 - \Delta_0^P$. Then Δ_P is a basis of $(\mathfrak{a}_P^G)^*$. For any root α, let $\alpha^\vee$ denote the corresponding co-root. The co-roots are elements of $\mathfrak{a}_0$ and form a basis of $\mathfrak{a}_0^G$. We can obtain a second basis of $(\mathfrak{a}_0^G)^*$ by taking the dual basis $\hat{\Delta}_0 = \{\beta_\alpha : \alpha \in \Delta_0\}$ of the basis $\Delta_0^\vee = \{\alpha^\vee : \alpha \in \Delta_0\}$. For a general parabolic P, $\hat{\Delta}_P = \{\beta_\alpha : \alpha \in \Delta_P\}$ defined similarly, is a second basis of $(\mathfrak{a}_P^G)^*$.

Given a parabolic subgroup $P = MN$, define $H_M : M_{\mathbf{A}} \to \mathfrak{a}_P$ by

$$e^{\langle H_M(m), \chi\rangle} = |\chi(m)| = \prod_v |\chi(m_v)|_v$$

for all $\chi \in X(M)$ and $m = \prod_v m_v \in M_{\mathbf{A}}$. Let K be a 'nice' maximal compact subgroup of $G_{\mathbf{A}}$ so that for any parabolic subgroup P, the Iwasawa decomposition holds: $G_{\mathbf{A}} = P_{\mathbf{A}} \cdot K$. We then have the *Langlands decomposition*: any $g \in G_{\mathbf{A}}$ can be written as $g = nmk$ where $n \in N_{\mathbf{A}}$, $m \in M_{\mathbf{A}}$ and $k \in K$. Using the Langlands decomposition H_M defined above can be extended to $H_P : G_{\mathbf{A}} \to \mathfrak{a}_P$.

For a parabolic subgroup P, let c_P be the *constant term operator*

$$c_P f(g) = \int_{N_{P,k}\backslash N_{P,\mathbf{A}}} f(ng)\, dn$$

for a left $N_{P,k}$-invariant, locally-L^1 function f. A function f is *cuspidal* if $c_P f = 0$ almost everywhere, for all proper parabolic subgroups P.

Fix a truncation parameter $0 < T \in \mathbf{R}$. For a parabolic subgroup P, let $\hat{\tau}_P$ denote the characteristic function of

$$\{H \in \mathfrak{a}_0 : \beta(H) > T \text{ for all } \beta \in \hat{\Delta}_P\}.$$

The *truncation* $\wedge^T\varphi$ of a function φ on $G_k\backslash G_{\mathbf{A}}$ is

$$\wedge^T\varphi(g) = \sum_P (-1)^{\dim A_P/A_G} \sum_{\gamma\in P_k\backslash G_k} \hat{\tau}_P(H_{P_0}(\gamma g))\, c_P\varphi(\gamma g)$$

where P runs over all parabolic subgroups. For each P, the truncated sum over $P_k\backslash G_k$ is finite. The truncation of a continuous function is of rapid decay on Siegel sets [Arthur 1980].

Let f be an automorphic form on $M_k\backslash M_{\mathbf{A}}$ where M is the Levi component of P and $\lambda \in (\mathfrak{a}_P^G)^*_{\mathbf{C}}$ (as in I.2.17, [Mœglin,Waldspurger 1995]). For $\lambda \in (\mathfrak{a}_P^G)^*_{\mathbf{C}}$ and $m \in M_{\mathbf{A}}$, let $\lambda(m) = e^{\langle H_M(m), \lambda\rangle}$. Define φ, attached to f and λ, on $G_{\mathbf{A}}$ by

$$\varphi(g) = \varphi(nmk) = \lambda(m) f(m) \delta_P^{1/2}(m)$$

where δ_P is the modulus function of P. The *Eisenstein series* induced from the parabolic P with data f, λ, is

$$E(g) = \sum_{\gamma\in P_k\backslash G_k} \varphi(\gamma g)$$

whenever the series converges. The Eisenstein series converges for all λ with real part in a positive open cone in $(\mathfrak{a}_P^G)^*$ depending on f, and the resulting $E(g)$ is an automorphic form on $G_k \backslash G_{\mathbf{A}}$ [Godement 1967].

Proposition. (II.1.7, [Mœglin,Waldspurger 1995]) Let $P = MN$ be a standard parabolic subgroup, f a spherical cuspform on $M_{\mathbf{A}}$ and $\lambda \in (\mathfrak{a}_P^G)^*_{\mathbf{C}}$. Then the constant term of E with respect to $R = M'N'$ is

$$c_R E_\lambda = \sum_w E^{M' \cap w^{-1} P w}(M(w)\varphi_\lambda)$$

where the sum is over

$$W(M, M') = \{w \in W : w^{-1}\alpha > 0 \text{ for } \alpha \in \Delta_0^R, wMw^{-1} \text{ is a standard Levi of } M'\}\,.$$

The operator $M(w)$ is the *intertwining operator*

$$M(w)f(g) = \int_{wP_k w^{-1} \cap N'_k \backslash N'_{\mathbf{A}}} f(w^{-1}ng)\, dn$$

2. Maass-Selberg relations

Maaß-Selberg relations give the inner products of two truncated Eisenstein series induced from cuspforms.

Theorem. *(Langlands) Let G be a reductive group and $P = MN$ a parabolic subgroup. Suppose that f, h are two cuspforms on M and $\lambda, \psi \in \mathfrak{a}^*_{P,\mathbf{C}}$. Let φ and φ' be the data attached to f, λ and h, ψ respectively. Let E_φ and $E_{\varphi'}$ be the Eisenstein series induced from these data on P. Then*

$$\langle \wedge^T E_\varphi, \wedge^T E_{\varphi'} \rangle \;=\; \sum_{Q, w_1, w_2} \mathrm{vol}(\mathfrak{a}_Q^G / L_Q)\, \frac{e^{(w_1\lambda + w_2\bar{\psi})(T)}}{\displaystyle\prod_{\alpha \in \Delta_Q} (w_1\lambda + w_2\bar{\psi})(\alpha^\vee)}\, \langle M(w_1, \lambda) f, M(w_2, \psi) h \rangle$$

where the sum is over the parabolic subgroups Q associate to P, $w_1 \in W(M_P, M_Q)$, $w_2 \in W(M_Q, M_P)$, and L_Q the lattice spanned by the co-roots corresponding to roots in Δ_Q.

3. The maximal parabolic subgroup case

Let $P = M_P N_P$ be a maximal parabolic subgroup in G. Then Δ_P consists of a single root α_P, which also is a basis of $(\mathfrak{a}_P^G)^*$. The co-root $\alpha_P^\vee$ is a basis of $\mathfrak{a}_P^G$. Let β_P be the dual to this basis which is the second basis of $(\mathfrak{a}_P^G)^*$. Therefore $(\mathfrak{a}_P^G)^*_{\mathbf{C}}$ can be identified with $\mathbf{C}$ by $s \to s\beta_P$.

For example, consider the case $G = GL_n$. Let P_0 be the upper triangular matrices in G and M_0 the diagonal matrices. The standard parabolic subgroups P in G consist of block upper-triangular matrices. Each P corresponds to a partition $[n_1, n_2, \ldots, n_r]$, $n_i > 0$, where

n_i is the size of the i-th block. The standard Levi component of P consists of block-diagonal matrices with corresponding sizes, and the unipotent subgroup consists of the elements with identity matrices of corresponding sizes on the diagonal.

In particular, a maximal parabolic subgroup of GL_n has two blocks on the diagonal. Let the sizes of the blocks be n_1, n_2, respectively. Then α_P is the following character in $(\mathfrak{a}_P^G)^*$:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \mapsto \frac{\det(m_1)^{1/n_1}}{\det(m_2)^{1/n_2}} .$$

The character β_P is $\beta_P = \alpha_P^{n_1 n_2/n}$.

Let $f \in \pi$ be a spherical cuspform on $M_{P,\mathbf{A}}$ with π an irreducible representation and $s \in \mathbf{C}$. Denote the kernel function attached to the data f, s by φ_s and the Eisenstein series induced from this data on P by E_s, suppressing the dependence on the cuspform f.

For $G = GL_n$, f corresponds to a pair of spherical cuspforms f_1, f_2 on $GL_{n_1}(\mathbf{A})$ and $GL_{n_2}(\mathbf{A})$ respectively. Then $f = f_1 \otimes f_2$ is the induced cuspform on $M_{P,\mathbf{A}}$ and φ_s attached to f, s is

$$\varphi_s(g) = \varphi_s(nmk) = f_1(m_1) f_2(m_2) s\beta_P(m) \delta_P(m)^{1/2} ,$$

where m_1, m_2 are the diagonal blocks of $m \in M_{P,\mathbf{A}}$.

Theorem. *The Eisenstein series E_s induced from cuspidal data on a not-self-associate maximal parabolic subgroup in a quasi-split reductive group G has no poles.*

Proof: Let P be a maximal *not* self-associate parabolic subgroup in G, i.e. that $W(M_P, M_P)$ is trivial. For $G = GL_n$, this is the case when the block sizes are different; $n_1 \neq n_2$. Let Q be the maximal parabolic associate to P and w the unique element in $W(M_P, M_Q)$. The Maaß-Selberg relations give

$$\begin{aligned} \langle \wedge^T E_s, \wedge^T E_s \rangle &= \mathrm{vol}(\mathfrak{a}_P^G / L_P) \, \frac{T^{(s\beta_P + \bar{s}\beta_P)(\alpha_P^\vee)}}{(s\beta_P + \bar{s}\beta_P)(\alpha_P^\vee)} \, \langle f, f \rangle \\ &\qquad + \mathrm{vol}(\mathfrak{a}_Q^G / L_Q) \, \frac{T^{(ws\beta_P + w\bar{s}\beta_P)(\alpha_Q^\vee)}}{(ws\beta_P + w\bar{s}\beta_P)(\alpha_Q^\vee)} \, \langle M(w)f, M(w)f \rangle \\ &= \mathrm{vol}(\mathfrak{a}_P^G / L_P) \left(\frac{T^{s+\bar{s}}}{s+\bar{s}} \, \langle f, f \rangle - \frac{T^{-s-\bar{s}}}{s+\bar{s}} \, \langle M(w)f, M(w)f \rangle \right) \end{aligned}$$

since $w\beta_P = -\beta_Q$ and hence $w\beta_P(\alpha_Q^\vee) = -1$. Using the intertwining operators and the continuation of the Eisenstein series, $M(w)f = c_f(s) f^w$ where f^w is the cuspform $f^w(m) = f(wmw^{-1})$ and $c_f(s)$ is a meromorphic function of $s \in (\mathfrak{a}_P^G)^*_{\mathbf{C}}$. Hence

$$\langle \wedge^T E_s, \wedge^T E_s \rangle = \|f\|^2 \frac{T^{2\,\mathrm{Re}\, s}}{2\,\mathrm{Re}\, s} - |c(s)|^2 \|f^w\|^2 \frac{T^{-2\,\mathrm{Re}\, s}}{2\,\mathrm{Re}\, s}$$

where the volume constant and dependence of the coefficient on f is suppressed.

The Eisenstein series has a pole exactly when one of the constant terms has a pole, and the order of the pole of the Eisenstein series is the same as that of the constant term with the highest order pole at that point. In the not-self-associate case, the constant term with respect to P is the kernel φ itself and it has no poles. The constant term with respect to Q is $M(w)\varphi_s$. Therefore the poles of E_s are the poles of $c_f(s)$ with the same order.

Let s_0 be a pole of E_s in the positive Weyl chamber $\operatorname{Re} s_0 > 0$ with order $m \geq 1$. Since s_0 is a pole of at least one of the constant terms, it is also a pole of $\wedge^T E_s$ of the same order. The real limit

$$\lim_{t\to 0}(it)^m \wedge^T E_{s_0+it}$$

is the leading term of the Laurent expansion of $\wedge^T E_s$ at $s = s_0$. Therefore the real limit

$$\lim_{t\to 0} t^{2m} \langle \wedge^T E_{s_0+it}, \wedge^T E_{s_0+it}\rangle$$

is the square of the norm of the leading term. Using the Maaß-Selberg relation this limit is (up to a positive constant) the square of the norm of the leading term of the Laurent expansion of $c(s)$ at $s = s_0$ times $-\|f^w\|^2 T^{-2\operatorname{Re} s_0}/(2\operatorname{Re} s_0)$, which is negative. This is impossible, so there are no poles in the positive Weyl chamber. There cannot be a pole at $\operatorname{Re} s = 0$ either, for the order of poles on both sides of the Maaß-Selberg relations do not match in that case. Using the functional equation of the Eisenstein series, we conclude that there are no poles anywhere.
□

Theorem. *The Eisenstein series E_s induced from cuspidal data on a self-associate maximal parabolic subgroup P does not have poles outside $(0, 1]$. The pole can occur only if the data is self-associate, i.e. if $w\pi = \pi$ for the non-trivial element $w \in W(M_P, M_P)$.*

Proof: In this case, the Maaß-Selberg relation is

$$\begin{aligned}\langle \wedge^T E_s, \wedge^T E_s\rangle &= \operatorname{vol}(\mathfrak{a}_P^G/L_P)\Big(\frac{T^{(s\beta_P+\bar{s}\beta_P)(\alpha_P^\vee)}}{(s\beta_P+\bar{s}\beta_P)(\alpha_P^\vee)}\langle f, f\rangle \\ &\quad + \frac{T^{(s\beta_P+w\bar{s}\beta_P)(\alpha_P^\vee)}}{(s\beta_P+w\bar{s}\beta_P)(\alpha_P^\vee)}\langle f, M(w)f\rangle + \frac{e^{(ws\beta_P+\bar{s}\beta_P)(\alpha_Q^\vee)}}{(ws\beta_P+\bar{s}\beta_P)(\alpha_Q^\vee)}\langle M(w)f, f\rangle \\ &\quad + \frac{T^{(ws\beta_P+w\bar{s}\beta_P)(\alpha_Q^\vee)}}{(ws\beta_P+w\bar{s}\beta_P)(\alpha_Q^\vee)}\langle M(w)f, M(w)f\rangle\Big)\end{aligned}$$

Using $w\beta_P = -\beta_P$ and suppressing the volume constant and f, the result simplifies to

$$\begin{aligned}\langle \wedge^T E_s, \wedge^T E_s\rangle &= \|f\|^2 \frac{T^{2\operatorname{Re} s}}{2\operatorname{Re} s} + \overline{c(s)}\langle f, f^w\rangle \frac{T^{2\operatorname{Im} s}}{2\operatorname{Im} s} \\ &\quad - c(s)\langle f^w, f\rangle \frac{T^{-2\operatorname{Im} s}}{2\operatorname{Im} s} - |c(s)|^2\|f^w\|^2 \frac{T^{-2\operatorname{Re} s}}{2\operatorname{Re} s}\end{aligned}$$

Let s_0 be a pole of E_s in the positive Weyl chamber $\operatorname{Re} s_0 > 0$ with order $m \geq 1$. As before, multiply both sides of the Maaß-Selberg relation evaluated at $s_0 + it$ by t^{2m} and let $t \to 0$. The left hand side is the square of the norm of the leading term of the Laurent expansion of $\wedge^T E_s$ at $s = s_0$. Except for the cases when $m = 1$, $\langle f, f^w \rangle \neq 0$, and $\operatorname{Im} s_0 = 0$, the first three terms on the right hand side have poles of order less than $2m$, so the right hand side will result in a negative value, giving a contradiction. The pole cannot be at $s = 0$, again because of differing orders of poles on the two sides of the Maaß-Selberg relation. Moreover, we know that the Eisenstein series converges for $s > 1$. These results altogether imply that the poles of E_s are simple and lie in $(0, 1]$. □

4. Singularities in higher rank parabolic subgroups

For parabolic subgroups of rank r greater than 1, $(\mathfrak{a}_P^G)^*_{\mathbf{C}}$ is identified with $\mathbf{C}^r$ using the basis $\hat{\Delta}_P$. The singularities of the Eisenstein series lie along hyperplanes in $\mathbf{C}^r$, and the residues are *multi-residues* along these hyperplanes, as explained below.

Let P be a parabolic subgroup. Given a root $\alpha \in \Delta_P$ and $c \in \mathbf{R}$, let

$$H(\alpha, c) = \{\Lambda \in \mathfrak{a}^*_{P,\mathbf{C}} : \Lambda(\alpha^\vee) = c\} .$$

An affine subspace $H \subset \mathfrak{a}^*_{P,\mathbf{C}}$ is *admissible* if H is an intersection of hyperplanes $H(\alpha, c)$. Consider meromorphic functions on $\mathfrak{a}^*_{P,\mathbf{C}}$ whose singularities lie along admissible subspaces. The residues of these functions are obtained as *multi-residues*.

Let $H_1 \supset H_2$ be two admissible subspaces of $\mathfrak{a}^*_{P,\mathbf{C}}$ with H_2 of codimension 1 in H_1. For a real unit vector v_1 in H_1 normal to H_2 and a meromorphic function f defined on H_1 whose singularities lie along admissible subspaces of H_1, the *residue of f along H_2* is

$$\operatorname{Res}_{H_2} f(\Lambda) = \frac{1}{2\pi i} \int_C f(\Lambda + \zeta v_1)\, d\zeta$$

where C is a sufficiently small, positively oriented simple curve around the origin. Then $\operatorname{Res}_{H_2} f(\Lambda)$ is a meromorphic function on H_2, well-defined up to sign, with singularities along admissible subspaces of H_2. Another way to obtain $\operatorname{Res}_{H_2} f(\Lambda)$ is as the limit $\lim_{\zeta \to 0} \zeta f(\Lambda + \zeta v_1)$ and in particular

$$\operatorname{Res}_{H_2} f(\Lambda) = \lim_{t \to 0} it f(\Lambda + itv_1)$$

with $t \in \mathbf{R}$.

In general, let

$$\mathfrak{a}^*_{P,\mathbf{C}} = H_n \supset H_{n-1} \supset \ldots \supset H_1 \supset H_0 = \{h\}$$

be a flag $\mathfrak{F}$ of admissible subspaces such that, for $1 \le i \le n$, H_{i-1} is of codimension 1 in H_i. For each i, let v_i be a unit normal in H_i to H_{i-1}. Let f be a meromorphic function on $\mathfrak{a}^*_{P,\mathbf{C}}$ with singularities along hyperplanes. Define inductively meromorphic functions

$$f_n = f,\ f_i = \mathrm{Res}_{H_i} f_{i+1},\ i = 0, \dots, n-1$$

The residue f_0 of f along $\mathfrak{F}$, denoted by $\mathrm{Res}_{\mathfrak{F}} f$, is well-defined up to a sign depending only on the choice of the unit normal vectors.

5. Rank-2 parabolic subgroups in GL_n

Let $G = GL_n$ and P a rank-2 parabolic subgroup of G corresponding to the partition $[n_1, n_2, n_3]$ with $n_1 + n_2 + n_3 = n$. There are exactly two maximal parabolic subgroups Q and R containing P: the first corresponds to the partition $[n_1, n_2 + n_3]$, the second to $[n_1 + n_2, n_3]$. Then Δ_P consists of two roots, α_Q and α_R, which also form a basis of $(\mathfrak{a}^G_P)^*$. The second basis of $(\mathfrak{a}^G_P)^*$ consists of β_Q and β_R, dual to the co-roots corresponding to the first basis elements. These elements are the pull-backs of the basis elements in the maximal parabolic subgroups Q and R. Identify $(\mathfrak{a}^G_P)^*_{\mathbf{C}}$ with $\mathbf{C}^2$ by $s = (s_1, s_2) \to s_1\beta_Q + s_2\beta_R$. For simplicity, we omit the β's when writing an element of $(\mathfrak{a}^G_P)^*_{\mathbf{C}}$.

Let $f = f_1 \otimes f_2 \otimes f_3 \in \pi$ be a spherical cuspform on $M_{P,\mathbf{A}}$ corresponding to the spherical cuspforms f_1, f_2 and f_3 on $GL_{n_1}(\mathbf{A}), GL_{n_2}(\mathbf{A})$ and $GL_{n_3}(\mathbf{A})$, respectively. Without loss of generality, assume π is irreducible. As before, for $s \in (\mathfrak{a}^G_P)^*_{\mathbf{C}}$ let φ_s denote the kernel attached to data f, s and E_s denote the Eisenstein series induced from the kernel φ_s. We investigate the singularities of E_s using the Maaß-Selberg relations.

Theorem. *Let P be a maximally self-associate rank-2 parabolic subgroup in $GL(n)$, i.e. $n_1 = n_2 = n_3$. Then the residues of Eisenstein series along a singular hyperplane are not square-integrable while the multi-residues are.*

Proof: The only parabolic subgroup associate to P is itself, hence the name maximally self-associate. The summation in the Maaß-Selberg relation is over pairs of Weyl group elements in $W(M_P, M_P)$. This group is generated by two elements: σ_1 and σ_2. The first exchanges the first two blocks and the second exchanges the last two blocks, acting on the β's by

$$\begin{array}{ll} \beta_Q \overset{\sigma_1}{\mapsto} \beta_R - \beta_Q \qquad & \beta_Q \overset{\sigma_2}{\mapsto} \beta_Q \\ \beta_R \overset{\sigma_1}{\mapsto} \beta_R & \beta_R \overset{\sigma_2}{\mapsto} \beta_Q - \beta_R \end{array}$$

The action then extends to all of $(\mathfrak{a}^G_P)^*_{\mathbf{C}}$ and $W(M_P, M_P)$ as in the following table:

w	id	σ_1	σ_2	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_1\sigma_2\sigma_1$
s	(s_1, s_2)	$(-s_1, s_1+s_2)$	$(s_1+s_2, -s_2)$	$(-s_1-s_2, s_1)$	$(s_2, -s_1-s_2)$	$(-s_2, -s_1)$

The Maaß-Selberg relation is (up to the volume constant)

$$\langle \wedge^T E_s, \wedge^T E_s \rangle = \sum_{w,w' \in W(M_P,M_P)} \frac{T^{(ws+w'\bar{s})(\alpha_Q^\vee)+(ws+w'\bar{s})(\alpha_R^\vee)}}{(ws+w'\bar{s})(\alpha_Q^\vee)\cdot(ws+w'\bar{s})(\alpha_R^\vee)} \langle M(w)f, M(w')f \rangle \ .$$

From the theory of the intertwining operators, $M(w)f$ is f^w times a product of coefficients from the maximal parabolic case. The product is over the positive roots α of M_P for which $w\alpha < 0$ and the coefficient corresponding to the root α takes the input $\langle ws, \alpha^\vee \rangle$. E.g. if $w = \sigma_1$, $M(w)f = c_{f_1 \otimes f_2}(s_1) f^w$ where $c_{f_1\otimes f_2}$ is the coefficient in the constant term of the Eisenstein series induced from data $f_1 \otimes f_2, s_1$ on the parabolic corresponding to the partition $[n_1, n_2]$ inside $GL(n_1+n_2)$. A few representative terms in the Maaß-Selberg relation are given in the table on the next page.

Using the results from the maximal parabolic case, we note that the singularities of the Eisenstein series in the rank-2 case occur only for $s_1, s_2 \in (0, 1]$.

Consider first the residue along a singular hyperplane. For a pole s_0 of the coefficients $c_{f_1\otimes f_2}$, $c_{f_2\otimes f_3}$ and $c_{f_1\otimes f_3}$ in the maximal parabolic cases, Eisenstein series in the rank-2 case has singularities along the root hyperplanes $H_1 = \{s : s_1 = s_0\}$, $H_2 = \{s : s_2 = s_0\}$ and $H_3 = \{s : s_1 + s_2 = s_0\}$ respectively.

Suppose the residue is along a singular hyperplane of the form H_1. For these s values, the terms in the Maaß-Selberg relation with singularities correspond to pairs (w, w') where w or w' includes σ_1 in its reduced expression. As in the maximal parabolic case, as $T \to \infty$, the terms with larger powers dominate on the right-hand side. In this case the residues with the largest power are

$$\frac{T^{s_0+2\,\mathrm{Re}\, s_2}}{2(2\,\mathrm{Re}\, s_2 + s_0)} \langle f^{\sigma_1}, f \rangle \,\mathrm{Res}_{s_1=s_0}\, c_{f_1\otimes f_2}(s_1)$$

and its conjugate. Therefore $\| \mathrm{Res} \wedge^T E_s \|^2$ behaves like

$$\frac{T^{s_0+2\,\mathrm{Re}\, s_2}}{2\,\mathrm{Re}\, s_2 + s_0} \mathrm{Re}(\langle f^{\sigma_1}, f \rangle \,\mathrm{Res}_{s_0}\, c_{f_1\otimes f_2}\,)$$

But as T approaches infinity, this term approaches infinity, proving that the residue is not square-integrable. This also shows that the real part of $\mathrm{Res}_{s_0}\, c_{f_1\otimes f_2}$ has to be positive.

Maaß-Selberg relations terms:

(w,w')	the corresponding term
(id,id)	$\frac{T^{2\operatorname{Re}s_1+2\operatorname{Re}s_2}}{2\operatorname{Re}s_1 2\operatorname{Re}s_2}\langle f,f\rangle$
(σ_1,id)	$\frac{T^{\overline{s_1}+2\operatorname{Re}s_2}}{-2\operatorname{Im}s_1(2\operatorname{Re}s_2+s_1)}c_{f_1\otimes f_2}(s_1)\langle f^{\sigma_1},f\rangle$
(σ_2,id)	$\frac{T^{2\operatorname{Re}s_1+\overline{s_2}}}{-(2\operatorname{Re}s_1+s_2)2\operatorname{Im}s_2}c_{f_2\otimes f_3}(s_2)\langle f^{\sigma_2},f\rangle$
$(\sigma_1\sigma_2,id)$	$\frac{T^{\overline{s_1}-2\operatorname{Im}s_2}}{(-2\operatorname{Im}s_1-s_2)(s_1+\overline{s_2})}c_{f_2\otimes f_3}(s_2)c_{f_1\otimes f_3}(s_1+s_2)\langle f^{\sigma_1\sigma_2},f\rangle$
$(\sigma_2\sigma_1,id)$	$\frac{T^{2\operatorname{Im}s_1+s_2}}{(s_1+\overline{s_2})(2\operatorname{Im}s_2-\overline{s_1})}c_{f_1\otimes f_2}(s_1)c_{f_1\otimes f_3}(s_1+s_2)\langle f^{\sigma_2\sigma_1},f\rangle$
$(\sigma_1\sigma_2\sigma_1,id)$	$\frac{T^{2\operatorname{Im}s_1+2\operatorname{Im}s_2}}{(s_1-\overline{s_2})(s_2-\overline{s_1})}c_{f_1\otimes f_2}(s_1)c_{f_2\otimes f_3}(s_2)c_{f_1\otimes f_3}(s_1+s_2)\langle f^{\sigma_1\sigma_2\sigma_2},f\rangle$
(σ_1,σ_1)	$\frac{T^{2\operatorname{Re}s_2}}{(-2\operatorname{Re}s_1)(2\operatorname{Re}s_1+2\operatorname{Re}s_2)}\lvert c_{f_1\otimes f_2}(s_1)\rvert^2\lVert f^{\sigma_1}\rVert^2$
(σ_1,σ_2)	$\frac{T^{s_2+\overline{s_1}}}{(-2\operatorname{Im}s_1+\overline{s_2})(2\operatorname{Im}s_2+s_1)}c_{f_1\otimes f_2}(s_1)\overline{c_{f_2\otimes f_3}(s_2)}\langle f^{\sigma_1},f^{\sigma_2}\rangle$
$\dots$	

Similarly, the residue along a hyperplane of the form H_2 or H_3 is not square-integrable.

Proposition. The maximal parabolic Eisenstein series E_s, $s\in\mathbf{C}$, for GL_n induced from self-associate data can have at most one pole in the region $(0,1]$.

Proof: (Proposition) At an intersection (s_0,s_0') of the singular hyperplanes, take a multi-residue along the flag

$$\{(s_0,s_0')\}\subset H_1\subset(\mathfrak{a}_P^G)^*_{\mathbf{C}}\,.$$

For $s_0\neq s_0'$, the only terms that survive taking residues twice have negative orders in T, hence vanish as T approaches infinity. □

Since there is at most one pole of the maximal parabolic case, the intersection of the singular hyperplanes can only occur if both the constant term coefficients corresponding to the simple roots have a pole. Take the flag

$$\{(s_0,s_0)\}\subset H_1\subset(\mathfrak{a}_P^G)^*_{\mathbf{C}}$$

to find the multi-residue. The only terms which do not vanish after taking the multi-residue are

$$|\operatorname{Res}_{s_0} c_{f_1 \otimes f_2}|\overline{c?}\ ,$$

its conjugate, and terms with T to a negative power. As $T \to \infty$, the limit approaches a finite value showing that the residue of the Eisenstein series along this flag is square-integrable. □

Theorem. *Let P be a not maximally self-associate rank-2 parabolic subgroup in $GL(n)$. Then the residues of Eisenstein series are not square-integrable.*

Proof: We do the case $n_1 = n_2 \neq n_3$. The cases $n_1 \neq n_2 = n_3$ and $n_1 = n_3 \neq n_2$ are similar. In this case $W(M_P, M_P)$ is generated only by σ_1 and the Maaß-Selberg relation involves only terms with w and w' either id or σ_1. There are four such terms. These terms, as in the residue along a singular hyperplane, blow up as the truncation parameter goes to infinity, showing that the residues of these not-maximally self-associate cases are not square-integrable. □

References

[Arthur 1978] James Arthur, "A trace formula for reductive groups I", *Duke Mathematical Journal* **45** (1978), pp. 911-952.

[Arthur 1980] James Arthur, "A trace formula for reductive groups II", *Compositio Mathematica* **40-1** (1980), pp. 87-121.

[Godement 1967] Roger Godement, *Introduction á la theorie de Langlands*, Séminaire Bourbaki 1966/1967, No. 321.

[Jacquet 1983] Herve Jacquet, "On the residual spectrum of GL_n", *Lie Group Representations II*, Lecture Notes in Mathematics, Vol 1041, Springer-Verlag, Berlin, 1984.

[Langlands 1966] Robert P. Langlands, "Eisenstein series", *Algebraic Groups and Discontinuous Subgroups*, Proceedings of Symposia in Pure Mathematics, Vol. 9, 1966.

[Langlands 1976] Robert P. Langlands, "On the Functional Equations Satisfied by Eisenstein Series", *Lecture Notes in Mathematics*, Vol. 544, Springer-Verlag, New York, 1976.

[Mœglin-Waldspurger 1989] Colette Mœglin, Jean-Loup Waldspurger, "Le spectre résiduel de $GL(n)$", *Ann. Sci. École Norm. Sup.*, **22** (1989), no.4, pp.605-674.

[Mœglin,Waldspurger 1995] Colette Mœglin, Jean-Loup Waldspurger, *Spectral Decomposition and Eisenstein Series: Une Paraphrase de l'Écriture*, Cambridge Tracts in Mathematics, No. 113, Cambridge University Press, 1995.

Department of Mathematics, Grand Valley State University, Allendale, MI 49401, e-mail: alayontf@gvsu.edu