# ADELIC APPROACH TO DIRICHLET $L$-FUNCTIONS

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Abstract. In the following I present the material of a one semester work done for my senior project under supervision of Okan Tekman. The work contains basic definitions and results needed to set the theory of zeta functions on adeles and ideles, and the results on zeta functions are used in the approach to Dirichlet $L$-function theory.

## Contents



## Introduction

The purpose of this senior project is to study the methods introduced by Tate in his thesis [T] to prove the analytic continuation of classical $\zeta$-functions and Dirichlet $L$-functions. Due to time limitation, we considered only the case of $\mathbb{Q}$ instead of an arbitrary number field. Although, in this case, the results can be easily obtained using the classical methods, our aim was to understand Tate's approach, and this is important for two reasons: On the one hand, the main tool in this approach is the local zeta functions, and the local computations are essentially the same in the case of an arbitrary number field as in the case of $\mathbb{Q}$. In fact, this is the real power of the method. On the other hand, even when restricted to $\mathbb{Q}$, Tate's work leads to a new way of looking at the classical modular functions and beyond [Gel].

In the first section, we define the field of $p$-adic numbers and consider their topological properties. In Section 2, we define unitary characaters of locally compact abelian groups. The Haar measure is introduced in Section 3 as a preparation for the next section, where the necessary tools from the Fourier analysis are reviewed. Section 5 contains the definition of local zeta functions and proves analytic continuation. Adeles and ideles are discussed in the sixth section, leading to the definition of the global zeta functions and their analytic continuation in Section 7. The next section contains the Poisson Summation Formula, the crucial ingredient of analytic continuation. The last section explains the connection between größencharacters and the Dirichlet characters, proving the analytic continuation of Dirichlet $L$-functions as a corollary of the analytic continuation of the global zeta functions. In this work, convergence results are generally omitted since they need a deeper examination and hence more time.

## 1. The field of $p$-adic numbers

A function $|\ | : \mathbb{Q} \to \mathbb{R}^+$ is called an *absolute value* on $\mathbb{Q}$ if for all $r, s \in \mathbb{Q}$
1) $|r| = 0$ iff $r = 0$.
2) $|rs| = |r||s|$.
3) $|r + s| \leq |r| + |s|$.

**Examples:**

**1)** The usual absolute value is an absolute value on $\mathbb{Q}$. We denote it by $|\ |_\infty$.

**2)** For a prime number $p$, we define $|\ |_p$, the $p$-adic absolute value of $r \in \mathbb{Q}$ as follows:

$$|r|_p = \begin{cases} p^{-n} & \text{if } r = p^n \cdot \dfrac{a}{b},\ (ab, p) = 1 \\ 0 & \text{if } r = 0\ . \end{cases}$$

In fact, $|\ |_p$ satisfies the non-Archimedean property:

$$|r + s|_p \leq \max\{|r|_p, |s|_p\}$$

which is stronger than (3).

Note that we have the following equality for $r \in \mathbb{Q}$:

$$\prod_{2 \leq p \leq \infty} |r|_p = 1\ .$$

Any absolute value $|\ |$ defines a metric on $\mathbb{Q}$ by $d(r, s) = |r - s|$ for $r, s \in \mathbb{Q}$. Two absolute values are said to be *equivalent* if they induce the same metric topology on $\mathbb{Q}$. By Ostrowski's Theorem we have: Every nontrivial absolute value $|\ |$ on $\mathbb{Q}$ is equivalent to $|\ |_p$ for some prime $p$ or $p = \infty$.

$\mathbb{Q}_p$ is defined to be the completion of $\mathbb{Q}$ with respect to $|\ |_p$. In particular, $\mathbb{Q}_\infty = \mathbb{R}$.

For the rest of this section, we assume $p < \infty$. We now define $\mathbb{Z}_p$, the *ring of p-adic integers* as

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p :\ |x|_p \leq 1\}$$

and $U_p$, the *group of p-adic units* as

$$U_p = \{x \in \mathbb{Q}_p :\ |x|_p = 1\}\ .$$

For $x \in \mathbb{Q}_p$, there exist unique $a_i \in \{0, 1, ..., p-1\}$ such that $x = \sum_{i=-\infty}^{\infty} a_i p^i$ with $a_i = 0$ for all $i \leq N$, for some $N$. This can be seen as follows: Let $x_n \in \mathbb{Q}$ and $\lim_{n\to\infty} x_n = x$. Then each $x_n$ has a finite $p$-adic expansion $\sum_i a_{ni} p^i$. Since $(x_n)$ is Cauchy, $(a_{ni})_n$ is eventually constant. If we let $a_i$ to be this constant, then $\sum_{i=-\infty}^{\infty} a_i p^i$ converges to $x$ in $\mathbb{Q}_p$.

Note that $|x|_p = \lim_{n\to\infty} |\sum_{i=-\infty}^{n} a_i p^i| = p^{-k}$ where $k = \min\{i :\ a_i \neq 0\}$. So for $x \in \mathbb{Z}_p, x = \sum_{i=0}^{\infty} a_i p^i$, and for $x \in U_p, x = \sum_{i=0}^{\infty} a_i p^i, a_0 \neq 0$. Any $x \in \mathbb{Q}_p$ can also be written as $x = p^{-k} \cdot u$ where $u \in U_p$.

Let us now consider the topology of $\mathbb{Q}_p$: We can take the balls of radius $p^{-n}$ around $a \in \mathbb{Q}_p$, $n \in \mathbb{N}$ as a neighborhood basis at $a$. $B(a, p^{-n}) = a + p^n\mathbb{Z}_p$ is both open and closed since $B(a, p^{-n}) = \{x \in \mathbb{Q}_p :\ |x - a|_p \leq p^{-(n+1)}\}$. $\mathbb{Q}_p$ is a group, so it will suffice to look only for the case $a = 0$ to understand the general case. $p^n\mathbb{Z}_p$'s are in fact compact. Being in a metric space with $p^n\mathbb{Z}_p$'s closed, we only need to show $p^n\mathbb{Z}_p$'s are totally bounded. We now take $\mathbb{Z}_p$ particularly. For a given $\epsilon > 0$, we have to find a finite cover of $\mathbb{Z}_p$ with $\epsilon$-balls. Observe that

$\mathbb{Z}_p = \cup_{m=0}^{p^k-1}(m+p^k\mathbb{Z}_p)$, so for $p^{-k} < \epsilon$, $\mathbb{Z}_p \subset \cup_{m=0}^{p^k-1} B(m,\epsilon)$. Similarly, $p^n\mathbb{Z}_p = \cup_{m=0}^{p^k-1} B(p^n m, p^n\epsilon)$. Now for any $a \in U \subset \mathbb{Q}_p$, $U$ open, we have $a \in a + p^n\mathbb{Z}_p \subset U$ for some $n$, and $a + p^n\mathbb{Z}_p$ is compact. So the condition for $\mathbb{Q}_p$ to be locally compact is satisfied. Furthermore, $\mathbb{Q}_p$ is totally disconnected. This follows from the fact that $a + p^n\mathbb{Z}_p$ is both open and closed for all $n$ and hence contains any connected set which contains $a$.

So we have:

**Theorem .** *$\mathbb{Q}_p$ is locally compact and totally disconnected.*

## 2. Characters

Let $G$ be a locally compact Hausdorff abelian group. A *(unitary) character* $\chi$ of $G$ is a continuous homomorphism $\chi : G \to \mathbb{C}^\times$ and $|\chi(g)| = 1$ for all $g \in G$.

We define $\hat{G}$ to be the set of all characters of $G$. $\hat{G}$ can be made into an abelian group by defining the product by pointwise multiplication. We define a topology on $\hat{G}$ by defining a neighborhood of the identity character $\mathbf{1}$ as $U(K,\epsilon) = \{\chi \in \hat{G} : \ |\chi(x) - 1| < \epsilon \text{ for } x \in K\}$, where $\epsilon > 0$ and $K$ is a compact subset of $G$.

**Theorem . 1)** *(Pontryagin Duality Theorem) The map $r : G \to \hat{\hat{G}}$ given by $r(g)(\chi) = \chi(g)$ for all $\chi \in \hat{G}$ is a topological isomorphism of groups.*
**2)** *If $G$ is compact, $\hat{G}$ is discrete. If $G$ is discrete, $\hat{G}$ is compact.*
**3)** *Let $H$ be a closed subgroup of $G$. Then $H^\perp = \{\chi \in \hat{G} : \ \chi(H) = \{1\}\}$ is a closed subgroup of $\hat{G}$ and we have*

$$\widehat{G/H} \cong H^\perp \text{ and } \hat{G}/H^\perp \cong \hat{H}\ .$$

**Examples:**

**1)** Let $G = \mathbb{Z}/n\mathbb{Z}$, the cyclic group of order $n$. For any $0 \le m < n$, there is a unique homomorphism $\chi_m : \mathbb{Z}/n\mathbb{Z} \to \mathbb{T}$ given by $\chi_m(1) = e^{2\pi i m/n}$. So we have $\hat{G} = \{\chi_m : \ 0 \le m < n\} \cong \mathbb{Z}/n\mathbb{Z}$. For any finite abelian group $G$, we can write $G$ as a direct sum of finitely many cyclic groups and $\widehat{G_1 \oplus \cdots \oplus G_k} \cong \hat{G}_1 \oplus \cdots \hat{G}_k$, we have $\hat{G} \cong G$.

**2)** Let $G = \mathbb{R}$. Define $\chi_\infty : \mathbb{R} \to \mathbb{C}^\times$ $,\chi_\infty(x) = e^{-2\pi i x}$. Then for any $y \in \mathbb{R}$, the homomorphism $x \mapsto \chi_\infty(yx)$ is a character of $\mathbb{R}$. Also for any character $\chi : \mathbb{R} \to \mathbb{C}^\times$, we can show that $\chi = \chi_\infty(y\cdot)$, for some $y \in \mathbb{R}$. Since $\chi(0) = 1$ and $\chi$ is continuous, for some $h > 0, c = \int_0^h \chi(t)\,dt \neq 0$. We can write

$$\chi(x)\int_0^h \chi(t)\,dt = \int_0^h \chi(x+t)\,dt = \int_x^{x+h} \chi(t)\,dt\ .$$

So we have

$$\chi'(x) = c^{-1}(\chi(x+h) - \chi(x)) = C\chi(x), \ \ C = c^{-1}(\chi(h) - 1)\ .$$

Hence $\chi(x) = e^{Cx}$ and since $|\chi(x)| = 1$, $C$ can be chosen as $C = 2\pi i y$, $y \in \mathbb{R}$. With this identification $\hat{\mathbb{R}} \cong \mathbb{R}$.

**3)** Let $G = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \cong \mathbb{R}/\mathbb{Z}$. We have $\hat{\mathbb{T}} \cong \widehat{\mathbb{R}/\mathbb{Z}} \cong \mathbb{Z}^\perp$. Now $\chi_\infty(yn) = 1$ for all $n \in \mathbb{N} \implies e^{-2\pi i y n} = 1$ for all $n \implies y \in \mathbb{Z}$. Hence $\mathbb{Z} \cong \hat{\mathbb{T}}$.

**4)** Let $G = \mathbb{Q}_p$. Define $\chi_p : G \to \mathbb{C}^\times$ by $\chi_p(x) = e^{2\pi i x}$. The function $\chi_p$ is to be taken as $\chi_p(x) = \exp(2\pi i \sum_{-k}^{-1} a_i p^i)$, if $x = \sum_{-k}^{\infty} a_i p^i$. To show that $\chi_p$ is in fact a continuous homomorphism we use:

$$\chi_p : \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Z}[1/p]/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z} \xrightarrow{\eta} \mathbb{C}^\times$$

where $\eta(x\mathbb{Z}) = e^{2\pi i x}$. In fact going from $\mathbb{Q}_p/\mathbb{Z}_p$ to $\mathbb{Z}[1/p]/\mathbb{Z}$ is the inverse of the natural homomorphism $\mathbb{Z}[1/p]/\mathbb{Z} \to \mathbb{Q}_p/\mathbb{Z} \to \mathbb{Q}_p/\mathbb{Z}_p$. The kernel of this homomorphism is $q + \mathbb{Z},\ q \in \mathbb{Z}_p \implies q \in \mathbb{Z}$. Also for any $x + \mathbb{Z}_p,\ x \in \mathbb{Q}_p$, we have $q + \mathbb{Z}_p = x + \mathbb{Z}_p$ where $q = \sum_{-k}^{-1} a_i p^i$ if $x = \sum_{-k}^{\infty} a_i p^i$ and also $q + \mathbb{Z} \mapsto q + \mathbb{Z}_p$ means the map is onto. Hence it is an isomorphism. So we will have $\chi_p$ is a homomorphism. $\chi_p$ will be continuous since its kernel $\mathbb{Z}_p$ is open in $\mathbb{Q}_p$.

For any $y \in \mathbb{Q}_p$, the homomorphism $\chi_p(y\cdot) : \mathbb{Q}_p \to \mathbb{C}^\times$ is a character of $\mathbb{Q}_p$. It can be easily seen that for $y_1 \neq y_2, y_1, y_2 \in \mathbb{Q}_p$, the characters $\chi_p(y_1\cdot)$ and $\chi_p(y_2\cdot)$ are different characters. So if we can show that all characters of $\mathbb{Q}_p$ are of the form $\chi_p(y\cdot)$ then we will have $\hat{\mathbb{Q}}_p \cong \mathbb{Q}_p$.

Let $\chi : \mathbb{Q}_p \to \mathbb{C}^\times$ be any character of $\mathbb{Q}_p$. Since $\chi$ is continuous, there exists a smallest $n \in \mathbb{Z}$ such that $\chi(p^n\mathbb{Z}_p) = \{1\}$. If there does not exist such $n$, then $\chi = \mathbf{1}$. We can consider, instead of $\chi$, $\chi' = \chi(p^{-n}y),\ y \in \mathbb{Q}_p$, which has $\chi'(\mathbb{Z}_p) = \{1\}$. So we may assume $n = 0$. $\chi$ will define a natural character on $p^{-1}\mathbb{Z}_p/\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$. We will then have $\chi(p^{-1}) = e^{2\pi i a_0/p}$ for some $a_0 \in \{1, ..., p-1\}$.

Assume we have $\chi(p^{-k}) = e^{2\pi i(a_0 p^{-k} + a_1 p^{-k+1} + \ldots + a_{k-1}p^{-1})}$ where $a_i \in \{1, ..., p-1\}$. Then

$$\chi(p^{-k-1})^p = \chi(p^{-k}) \implies \chi(p^{-k-1}) = e^{2\pi i(a_0 p^{-k-1} + a_1 p^{-k} + \ldots + a_{k-1}p^{-2})}\, e^{2\pi i a_k/p}$$

for some $a_k \in \{1, ..., p-1\}$. Let $y = \sum_{i=0}^{\infty} a_i p^i$. Then $|y| = 1$ and $\chi(p^{-k}) = \chi_p(yp^{-k})$ for all $k > 0$. So $\chi(x) = \chi_p(yx)$ for all $x \in \mathbb{Q}_p$.

Also one can show that $\mathbb{Q}_p$ is topologically isomorphic to $\hat{\mathbb{Q}}_p$. A neighborhood of $\mathbf{1}$ in $\hat{\mathbb{Q}}_p$ is $\{y \in \mathbb{Q}_p :\ |\chi_p(yx) - 1| < \frac{1}{n} \text{ for } |x| \leq p^m\},\ n > 0,\ m \in \mathbb{Z}$. It follows that $\chi_p(y\cdot)$ is in this neighborhood if and only if $|y| \leq p^{-m}$.

The reason for choosing the character on $\mathbb{R}$ as $\chi_\infty(x) = e^{-2\pi i x}$ while we chose the characters on $\mathbb{Q}_p$ as $\chi_p(x) = e^{2\pi i x}$ is to have the following equality:

$$\prod_p \chi_p(r) = 1 \ \text{ for } r \in \mathbb{Q}\ .$$

This equality follows if we write $r = \sum_p \dfrac{a_p}{p^{n_p}} + m$, where $a_p, n_p, m \in \mathbb{Z}$. Then $\chi_p(r) = e^{2\pi i a_p/p^{n_p}}$ and $\chi_\infty(r) = e^{-2\pi i \sum_p a_p/p^{n_p}}$.

**5)** Let $G = \mathbb{R}^\times$. We can write $\mathbb{R}^\times \cong \{-1, 1\} \times \mathbb{R}^+$. From this representation we can deduce that any character $\chi$ of $\mathbb{R}^\times$ is of the form $\chi(x) = \mathrm{sgn}(x)|x|^s$ or $\chi(x) = |x|^s$.

**6)** Let $G = {\mathbb{Q}_p}^\times$. We can write ${\mathbb{Q}_p}^\times \cong <p> \times U_p$. First let us consider the characters of $U_p$. If $\chi : U_p \to \mathbb{C}^\times$, there exists $n > 0$ such that $\chi(1 + p^n\mathbb{Z}_p) = \{1\}$. Let us denote $1 + p^n\mathbb{Z}_p$ by $U_p^{(n)}$. $U_p$ is $U_p^{(0)}$ by convention. If $n$ is the smallest of such $n$'s, $p^n$ is called the *conductor* of $\chi$. $\chi$ is called *unramified* if the conductor is 1. Now $U_p^{(n)}$ is in the kernel of $\chi$. So $\chi$ defines a character of $U_p/U_p^{(n)}$, which is finite.

Characters of $<p>$ which are trivial on $U_p$ are of the form $\chi : {\mathbb{Q}_p}^\times \to \mathbb{C}^\times,\ \chi(x) = |x|_p^s$.

So we get that any character $\chi : {\mathbb{Q}_p}^\times \to \mathbb{C}^\times$ is of the form $\chi(x) = |x|_p^s w(x|x|_p)$, where $w$ is a character of $U_p$.

## 3. Haar Measure

Let $G$ be a locally compact Hausdorff group. A *left* (resp. *right* ) *Haar measure* on $G$ is a nonzero regular Borel measure $\mu$ on $G$ satisfying $\mu(gE) = \mu(E)$ (resp. $\mu(Eg) = \mu(E)$) for every Borel set $E \subset G$ and $g \in G$. In an abelian group $G$ a left Haar measure is also a right Haar measure.

**Theorem .** *Every locally compact Hausdorff abelian group $G$ has a Haar measure. Furthermore this measure is unique up to a constant, i.e. if $\mu$, $\lambda$ are two Haar measures on $G$, there exists $0 < c < \infty$ such that $\mu = c\lambda$.*

Let $G$ be abelian. $\mu$ is a Haar measure on $G$ if and only if

$$\int_G f(hg)\, dg = \int_G f(g)\, dg$$

for all positive continuous functions with compact support (i.e. $f(x) = 0$ for $x \notin K$, $K$ compact in $G$) and $h \in G$.

**Examples:**

**1)** The usual Lebesgue measure is a Haar measure on $\mathbb{R}$ (as an additive group), since

$$\int_{\mathbb{R}} f(x+h)\, dx = \int_{\mathbb{R}} f(x)\, dx$$

for all $f$ continuous and $h \in \mathbb{R}$.

**2)** On $\mathbb{R}^\times$, $\dfrac{dx}{|x|}$ is a Haar measure, since

$$\int_{\mathbb{R}^\times} f(hx)\frac{dx}{|x|} = \int_{\mathbb{R}^\times} f(hh^{-1}x)\frac{d(h^{-1}x)}{|h^{-1}x|} = \int_{\mathbb{R}^\times} f(x)\frac{dx}{|x|}\ .$$

**3)** On $\mathbb{Q}_p$, we know that, by theorem, there exists a Haar measure $\mu$. We can normalize $\mu$ by letting $\mu(\mathbb{Z}_p) = 1$. We can write for $n \geq 0, \mathbb{Z}_p = \cup_{i=0}^{p^n-1} i + p^n\mathbb{Z}_p$, where $i + p^n\mathbb{Z}_p$'s are disjoint for different $i$'s. So $\mu(\mathbb{Z}_p) = p^n\mu(p^n\mathbb{Z}_p)$ which implies $\mu(p^n\mathbb{Z}_p) = p^{-n}$. Similarly, for $n < 0, p^n\mathbb{Z}_p = \cup_{i=0}^{p^{-n}-1} ip^n + \mathbb{Z}_p$, and hence $\mu(p^n\mathbb{Z}_p) = p^{-n}$.

**4)** On ${\mathbb{Q}_p}^\times$, $\dfrac{dx}{|x|_p}(1 - p^{-1})^{-1}$ is a Haar measure where $dx$ is the unique normalized Haar measure on $\mathbb{Q}_p$. Given $a \in {\mathbb{Q}_p}^\times$ with $|a|_p = p^n, \mu(a\mathbb{Z}_p) = \mu(p^n\mathbb{Z}_p) = p^{-n}$. Similarly, $\mu(ap^n\mathbb{Z}_p) = |a|_p\mu(p^n\mathbb{Z}_p)$ for all $a \in {\mathbb{Q}_p}^\times, n \in \mathbb{Z}$. So we have

$$\int_{{\mathbb{Q}_p}^\times} f(ax)\frac{dx}{|x|_p} = \int_{{\mathbb{Q}_p}^\times} f(x)\frac{dx}{|x|_p}$$

for any step function $f$ and this suffices for the proof of the statement for any $f$.

Let $G$ be a locally compact Hausdorff abelian froup, $H$ be a closed subgroup of $G$. Then $G/H$ is a locally compact Hausdorff abelian group.

**Theorem** (Fubini's Theorem)**.** *Let $\mu_G$ and $\mu_H$ be Haar measures on $G$ and $H$. Then the Haar measure on $G/H$ can be chosen in such a way that*

$$\int_G f(g)\ dg = \int_{G/H}\left(\int_H f(gh)\ dh\right) d(gH)\ .$$

*for all compactly supported continuous functions $f$ on $G$ .*

## 4. The Fourier Transformation

Let $G$ be a locally compact abelian group, $\hat{G}$ be the character group of $G$. Choose Haar measures on $G$ and $\hat{G}$. For a continuous function $f : G \to \mathbb{C}$ such that $\int_G |f(g)|\, dg < \infty$, we define the *Fourier transform* of $f$, $\hat{f} : \hat{G} \to \mathbb{C}$ as

$$\hat{f}(\chi) = \int_G \overline{\chi(g)} f(g)\, dg\ .$$

**Theorem** (Fourier Inversion Theorem)**.** *If $f \in L^1(G)$, $f$ is continuous and $\hat{f} \in L^1(\hat{G})$, we have*

$$f(g) = \int_{\hat{G}} \chi(g)\hat{f}(\chi)\, d\chi$$

*if Haar measure $d\chi$ on $\hat{G}$ is suitably normalized relative to the given Haar measure $dg$ on $G$.*

The normalized measure on $\hat{G}$ is called the *dual measure* of $dg$.

**Examples:**

**1)** Let $G$ be a finite abelian group. Then $\hat{f} = \sum_{g\in G} \overline{\chi(g)} f(g)$ is the Fourier transform of $f$ and the dual of counting measure on $G$ is the dual measure divided by $|G|$ on $\hat{G}$.
***Proof:*** Let $S = \sum_{g\in G} \chi(g)$ for $\chi \in \hat{G}$. Then

$$\chi(h)S = \sum_{g\in G} \chi(h)\chi(g) = \sum_{g\in G} \chi(hg) = \sum_{g\in G} \chi(g) = S\ .$$

It follows that

$$\sum_{g\in G} \chi(g) = \begin{cases} |G| & \text{if } \chi = \mathbf{1}, \\ 0 & \text{if } \chi \neq \mathbf{1}\ . \end{cases}$$

Applying this formula to $\hat{G}$ and $\hat{\hat{G}} \cong G$, we get

$$\sum_{\chi\in \hat{G}} \chi(g) = \begin{cases} |G| & \text{if } g = e\ , \\ 0 & \text{if } g \neq e\ . \end{cases}$$

Or in a different form, using $\chi(h^{-1}) = \overline{\chi(h)}$, we can write

$$\frac{1}{|G|} \sum_{\chi\in\hat{G}} \overline{\chi(h)}\chi(g) = \delta_h(g)$$

for all $g \in G$ where $\delta_h(g) = \begin{cases} 1 & \text{if } g = h\ , \\ 0 & \text{if } g \neq h\ . \end{cases}$

The trivial equality $f(g) = \sum_{h\in G} f(h)\delta_h(g)$ gives

$$f(g) = \frac{1}{|G|} \sum_{\chi\in\hat{G}} \left( \sum_{h\in G} \overline{\chi(h)} f(h) \right) \chi(g) = \frac{1}{|G|} \sum_{\chi\in\hat{G}} \hat{f}(\chi)\chi(g)$$

for all $g \in G$. $\square$

**2)** Let $G = \mathbb{R}$. For any $f \in L^1(\mathbb{R})$, the Fourier transform of $f$ is $\hat{f}(y) = \int_{\mathbb{R}} e^{2\pi i y x} f(x)\, dx$.

The Lebesgue measure on $\hat{\mathbb{R}} \cong \mathbb{R}$ is the dual of the Lebesgue measure on $\mathbb{R}$.
***Proof:*** We had the identification of $\hat{\mathbb{R}}$ as $\chi \in \hat{\mathbb{R}}, \chi = \chi_\infty(y\cdot)$ for some $y \in \mathbb{R}$. So

$$\hat{f}(y) = \int_{\mathbb{R}} \overline{\chi_\infty(yx)} f(x)\, dx = \int_{\mathbb{R}} e^{2\pi i yx} f(x)\, dx \,.$$

Now to show that the Lebesgue measure is self-dual on $\mathbb{R}$, notice that

$$f(g) = c \int_{\hat{G}} \chi(g) \hat{f}(\chi)\, d\chi$$

for some $0 < c < \infty$ and the measure on $\hat{G}$ is dual to the measure on $G$ if $c = 1$. So if we can show that $c = 1$ for some particular $f$, we will be done.

Let $f = e^{-\pi x^2}$. Then

$$\hat{f}(y) = \int_{\mathbb{R}} e^{2\pi i yx} e^{-\pi x^2}\, dx = \int_{\mathbb{R}} e^{-\pi(x-iy)^2} e^{-\pi y^2}\, dx = e^{-\pi y^2} \int_{\mathbb{R}} e^{-\pi x^2}\, dx = e^{-\pi y^2} \,,$$

and similarly

$$\int_{\hat{\mathbb{R}}} \chi(x) \hat{f}(\chi)\, d\chi = \int_{\mathbb{R}} e^{-2\pi i yx} e^{-\pi y^2}\, dy = e^{-\pi x^2} = f(x) \,.$$

□

**3)** If $G = \mathbb{T}$, we showed that $\hat{G} = \mathbb{Z}$. The Fourier transform of $f : \mathbb{T} \to \mathbb{C}^\times$ is given by $\hat{f} : \mathbb{Z} \to \mathbb{C}^\times$, $\hat{f}(n) = \int_0^1 e^{-2\pi i n x} f(x)\, dx$. The inverse Fourier transform will be given as $f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$. Hence the Lebesgue measure on $\mathbb{T}$ and the counting measure on $\mathbb{Z}$ will be dual.

**4)** Let $G = \mathbb{Q}_p, f = \mathrm{char}(\mathbb{Z}_p)$. The Fourier transform of $f$ is

$$\hat{f}(y) = \int_{\mathbb{Q}_p} \overline{\chi_p(yx)} f(x)\, dx = \int_{\mathbb{Z}_p} \overline{\chi_p(yx)}\, dx = \begin{cases} 1 & \text{if } y \in \mathbb{Z}_p, \\ 0 & \text{if } y \notin \mathbb{Z}_p \end{cases}$$

since $\chi_p(y\cdot)$ is also a character on $\mathbb{Z}_p$ and its integral is $\mu(\mathbb{Z}_p) = 1$ if $\chi_p(y\cdot)$ is $\mathbf{1}$ on $\mathbb{Z}_p$ and 0 if not. So $\hat{f} = \mathrm{char}(\mathbb{Z}_p)$. Since $\hat{G} \cong \mathbb{Q}_p$, in the same way we have $f(x) = \int_{\mathbb{Q}_p} \hat{f}(y) \chi_p(yx)\, dy$. This shows that the Haar measure on $\mathbb{Q}_p$ is dual to itself.

## 5. Local Zeta Functions

A function $f : \mathbb{R} \to \mathbb{C}$ is called a *Schwartz function* on $\mathbb{R}$ if $f \in C^\infty(\mathbb{R})$ and $|p(x) f^{(n)}(x)| \to 0$ as $|x| \to \infty$ for all $n \geq 0$ and all polynomials $p(x)$. $e^{-x^2}$ is an example of a Schwartz function on $\mathbb{R}$. A *Schwartz function* on $\mathbb{Q}_p$ is a function $f : \mathbb{Q}_p \to \mathbb{C}$ such that $f$ is locally constant and has compact support. $f$ is *locally constant* means for each $a \in \mathbb{Q}_p$, there exists $n$ such that $f$ is constant on $a + p^n \mathbb{Z}_p$. $f$ has *compact support* means the closure of $\{x :\ f(x) \neq 0\}$ is compact. If f is a Schwartz function on $\mathbb{Q}_p$, then there exist $n \in \mathbb{Z}$ and $a_i \in \mathbb{Q}_p,\ c_i \in \mathbb{C},\ i = 1, ..., k$ such that $f = \sum_{i=1}^k c_i\, \mathrm{char}(a_i + p^n \mathbb{Z}_p)$. The space of Schwartz functions on $\mathbb{Q}_p$ is denoted by $\mathcal{S}(\mathbb{Q}_p)$.

Let $f$ be a Schwartz function, $c$ be a character of ${\mathbb{Q}_p}^\times$. A *local zeta function* on $\mathbb{Q}_p$ is a function of a complex variable $s$ defined by

$$\zeta(f, c, s) = \int_{{\mathbb{Q}_p}^\times} f(x) c(x) |x|_p^s\, d^\times x \,.$$

This integral converges absolutely and uniformly on compact subsets of $\{s \in \mathbb{C} :\ \mathrm{Re}\, s > 0\}$, hence defines an analytic function for these $s$.

**Proposition .** Let $f_1, f_2 \in \mathcal{S}(\mathbb{Q}_p)$ and let $c$ be a character of ${\mathbb{Q}_p}^\times$. Then for $0 < \mathrm{Re}\, s < 1$,

$$\zeta(f_1, c, s)\; \zeta(\hat{f}_2, c^{-1}, 1-s) = \zeta(\hat{f}_1, c^{-1}, 1-s)\; \zeta(f_2, c, s)\ .$$

***Proof:***

$$\begin{aligned}
\zeta(f_1, c, s)\; \zeta(\hat{f}_2, c^{-1}, 1-s) &= \int_{{\mathbb{Q}_p}^\times} \int_{{\mathbb{Q}_p}^\times} f_1(x)c(x)|x|_p^s\; \hat{f}_2(y)c^{-1}(y)|y|_p^{1-s}\; d^\times x d^\times y \\
&= \int_{{\mathbb{Q}_p}^\times} \int_{{\mathbb{Q}_p}^\times} f_1(x)c(x)|x|_p^s\; \hat{f}_2(xy)c^{-1}(xy)|xy|_p^{1-s}\; d^\times x d^\times y \\
&= \int_{{\mathbb{Q}_p}^\times} \int_{{\mathbb{Q}_p}^\times} |x|_p|y|_p^{1-s} f_1(x)\hat{f}_2(xy)c^{-1}(y)\; d^\times x d^\times y \\
&= \int_{{\mathbb{Q}_p}^\times} \int_{{\mathbb{Q}_p}^\times} \int_{\mathbb{Q}_p} |x|_p|y|_p^{1-s} f_1(x) f_2(z)\overline{\chi_p(xyz)}c^{-1}(y)\; dz d^\times x d^\times y \\
&= \int_{{\mathbb{Q}_p}^\times} |y|_p^{1-s}c^{-1}(y) \left( \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} f_1(x) f_2(z)\overline{\chi_p(xyz)}\; dz dx \right) d^\times y \\
&= \int_{{\mathbb{Q}_p}^\times} |y|_p^{1-s}c^{-1}(y) \left( \int_{{\mathbb{Q}_p}^\times} \hat{f}_1(yz) f_2(z)|z|_p\; d^\times z \right) d^\times y \\
&= \zeta(f_2, c, s)\zeta(\hat{f}_1, c^{-1}, 1-s)\ . \qquad \Box
\end{aligned}$$

**Theorem .** *A local zeta function possesses an analytic continuation to the whole complex plane as a meromorphic function of $s$. Moreover, there exists a meromorphic function $\rho(c,s)$ such that*

$$\zeta(f, c, s) = \rho(c, s)\zeta(\hat{f}, c^{-1}, 1-s)$$

*for all $s$.*

The proof of the theorem follows from the proposition and the following examples.

**Examples:**

**1)** Let $p = \infty,\ f(x) = e^{-\pi x^2},\ c = \mathbf{1}$. Then

$$\begin{aligned}
\zeta(f, c, s) &= \int_{\mathbb{R}^\times} e^{-\pi x^2}|x|^s\; d^\times x = 2\int_0^\infty e^{-\pi x^2} x^{s-1}\; dx \\
&= 2\int_0^\infty e^{-u}(\frac{u}{\pi})^{\frac{s-1}{2}} \frac{du}{2\pi(\frac{u}{\pi})^{\frac{1}{2}}} = \frac{1}{\pi^{\frac{s}{2}}} \int_0^\infty e^{-u}u^{\frac{s}{2}-1}\; du \\
&= \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\ .
\end{aligned}$$

$$\zeta(\hat{f}, c^{-1}, 1-s) = \pi^{\frac{s-1}{2}}\Gamma(\frac{1-s}{2})\ .$$

$$\rho(c, s) = \frac{\zeta(f, c, s)}{\zeta(\hat{f}, c^{-1}, 1-s)} = \pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\ .$$

**2)** Let $p = \infty, f(x) = xe^{-\pi x^2}, c(x) = \mathrm{sgn}(x)$.

$$\zeta(f, c, s) = \pi^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2})\ .$$

$$\hat{f}(y) = iye^{-\pi y^2}\ .$$

$$\zeta(\hat{f}, c^{-1}, 1-s) = i\pi^{-\frac{(1-s)+1}{2}}\Gamma\left(\frac{(1-s)+1}{2}\right) .$$

$$\rho(c,s) = -i\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s+1}{2})}{\Gamma(\frac{(1-s)+1}{2})} .$$

**3)** Let $p$ be prime, $f = \mathrm{char}(\mathbb{Z}_p)$, $c$ such that conductor of $c = 1$.

$$\begin{aligned}
\zeta(f,c,s) &= \int_{\mathbb{Q}_p{}^\times} \mathrm{char}(\mathbb{Z}_p)(x)c(x)|x|_p^s \, d^\times x = \int_{\mathbb{Z}_p} c(x)|x|_p^s \, d^\times x \\
&= \sum_{j=0}^{\infty} \int_{p^j U_p} c(x)|x|_p^s \, d^\times x = \sum_{j=0}^{\infty} p^{-js}c(p)^j \int_{U_p} d^\times x \\
&= \sum_{j=0}^{\infty} (c(p)p^{-s})^j = \frac{1}{1-p^{-s}c(p)} .
\end{aligned}$$

$$\zeta(\hat{f}, c^{-1}, 1-s) = \frac{1}{1-p^{s-1}c^{-1}(p)} \quad \text{and} \quad \rho(c,s) = \frac{1-p^{s-1}c^{-1}(p)}{1-p^{-s}c(p)} .$$

**4)** Let $p$ be prime, $f = \mathrm{char}(1+p^n\mathbb{Z}_p)$, $c$ such that conductor of $c = p^n$.

$$\zeta(f,c,s) = \int_{\mathbb{Q}_p{}^\times} \mathrm{char}(1+p^n\mathbb{Z}_p)(x)c(x)|x|_p^s \, d^\times x = \int_{U_p^{(n)}} d^\times x .$$

$$\begin{aligned}
\hat{f}(y) &= \int_{\mathbb{Q}_p} \overline{\chi_p(yx)}\, \mathrm{char}(1+p^n\mathbb{Z}_p)(x) \, dx = \int_{U_p^{(n)}} \overline{\chi_p(yx)} \\
&= \overline{\chi_p(y)} \int_{p^n\mathbb{Z}_p} \overline{\chi_p(yu)} \, du = \overline{\chi_p(y)} p^{-n} \, \mathrm{char}(p^{-n}\mathbb{Z}_p)(y) .
\end{aligned}$$

$$\begin{aligned}
\zeta(\hat{f}, c^{-1}, 1-s) &= \int_{\mathbb{Q}_p{}^\times} \overline{\chi_p(x)} p^{-n} \, \mathrm{char}(p^{-n}\mathbb{Z}_p) c^{-1}(x)|x|_p^{1-s} \, d^\times x \\
&= p^{-n} \int_{p^{-n}\mathbb{Z}_p} \overline{\chi_p(x)c(x)}|x|_p^{1-s} \, d^\times x \\
(*) \qquad &= p^{-n} \sum_{j=-n}^{\infty} \int_{p^j U_p} \overline{\chi_p(x)c(x)}|x|_p^{1-s} \, d^\times x \\
&= p^{-n} \sum_{j=-n}^{\infty} \overline{c(p)}^j p^{-j(1-s)} \int_{U_p} \overline{\chi_p(p^j u)c(u)} \, d^\times u .
\end{aligned}$$

Now for $j \geq 0$, we have

$$\int_{U_p} \overline{\chi_p(p^j u)c(u)} \, d^\times u = \int_{U_p} \overline{c(u)} \, d^\times u = 0$$

since $c$ is non-trivial on $U_p$.

For $-n < j < 0$, we have

$$\begin{aligned}\int_{U_p} \overline{\chi_p(p^j u)c(u)}\, d^\times u &= \sum_{x \in U_p/U_p^{(n-1)}} \int_{xU_p^{(n-1)}} \overline{\chi_p(p^j u)c(u)}\, d^\times u \\ &= \sum_x \overline{\chi_p(p^j x)c(x)} \int_{U_p^{(n-1)}} \overline{c(u')}\, d^\times u' = 0\end{aligned}$$

since again $c$ is non-trivial on $U_p^{(n-1)}$.

Hence the equation $*$ becomes

$$\begin{aligned}\zeta(\hat{f}, c^{-1}, 1-s) &= p^{-n} p^{n(1-s)} \sum_{x \in U_p/U_p^{(n)}} \overline{\chi_p(p^{-n}x)c(p^{-n}x)} \int_{U_p^{(n)}} d^\times u \\ &= p^{n(\frac{1}{2}-s)} \overline{\rho(c)} \int_{U_p^{(n)}} d^\times u = p^{n(\frac{1}{2}-s)} \overline{\rho(c)} \zeta(f, c, s)\end{aligned}$$

where $\rho(c) = \sum_{x \in U_p/U_p^{(n)}} \chi_p(p^{-n}x)c(p^{-n}x)p^{-n/2}$. $\rho(c)$ is called a *generalized Gauss sum*. The functional equation implies that $|\rho(c)| = 1$. Hence

$$\rho(c, s) = p^{n(s-\frac{1}{2})} \rho(c) \ .$$

## 6. Adeles and Ideles

Let $\{(G_p, H_p)\}_p$ be a collection of locally groups $G_p$ and for almost all $p$, open compact subgroups $H_p$ of $G_p$. The expression "almost all $p$" will be used as a short definition for "all but a finite number of $p$'s". The *restricted direct product* of $\{(G_p, H_p)\}_p$ is the set

$$G = \{(g_p)_p :\ g_p \in H_p \text{ for almost all } p\} \ .$$

$G$ is a group under componentwise multiplication. We topologize it be defining the open sets aroung identity element $(e_p)_p \in G$ as

$$V = \prod_p U_p$$

where $U_p \subset G_p$ open and $U_p = H_p$ for almost all $p$.

If $S$ is a finite set contaning the $p$'s for which $H_p$ is not defined, $G^S = \prod_{p \notin s} H_p \times \prod_{p \in S} G_p$ is an open subgroup of $G$. The topology on $G$ is defined as the weakest topology to make $G^S$ an open subgroup for any finite set, $S$.

Now let $G_p = \mathbb{Q}_p$ and for $p \neq \infty$, let $H_p = \mathbb{Z}_p$ as additive groups. Then the restricted direct product is called *adeles* and is denoted by $\mathbb{A}$. We can embed $\mathbb{Q}$ in $\mathbb{A}$ via the diagonal map $q \mapsto (q, q, ...) \in \mathbb{A}$ since if $q = \dfrac{a}{q} \in \mathbb{Q},\ q \in \mathbb{Z}_p$ for $p \not| \, b$.

**Proposition .** $\mathbb{Q} \subset \mathbb{A}$ is discrete.

***Proof:*** It will suffice to show that there exists a neighborhood of 0 which contains no other element of $\mathbb{Q}$. Define

$$O = (-1/2, 1/2) \times \prod_p \mathbb{Z}_p \ .$$

$O$ is open on $\mathbb{A}$ and $0 \in O$. Suppose $q \in \mathbb{Q} \cap O$. Then $q \in \mathbb{Z}_p$ for all $p$, which implies $q \in \mathbb{Z}$ and since $-1/2 < q < 1/2,\ q = 0$. So $O$ doesn't contain an element of $\mathbb{Q}$ other than 0. $\square$

**Proposition .** $\mathbb{A}/\mathbb{Q}$ is compact.

***Proof:*** If we show that there exists $D \subset \mathbb{A}$, $D$ compact and $D + \mathbb{Q} = \mathbb{A}$, then we will be done. Let $D = [0,1] \times \prod_p \mathbb{Z}_p$. $D$ is compact. Now take and $(x_p)_p \in \mathbb{A}$. Then $x_p \in \mathbb{Z}_p$ for $p \notin S$, $S$ a finite set of $p$'s. For $p \in S, x_p = \dfrac{n_p}{p^{k_p}} + z_p$, where $z_p \in \mathbb{Z}_p, n_p, k_p \in \mathbb{Z}^+$. Let $r$ be such that

$$r = \sum_{p \in S} \frac{n_p}{p^{k_p}} \ .$$

Then $x_p - r \in \mathbb{Z}_p$ for all $p$. Let $r_0 = [\![ x_\infty - r ]\!]$. Then $x_p - r - r_0 \in \mathbb{Z}_p$ for all $p$ and $x_\infty - r - r_0 \in [0,1]$. Hence $(x_p - r - r_0)_p \in D$. Hence we have $D + \mathbb{Q} = \mathbb{A}$, which means $D$ maps onto $\mathbb{A}/\mathbb{Q}$ by the natural homomorphism. Hence $\mathbb{A}/\mathbb{Q}$ is compact. $\square$ Let $G = {\mathbb{Q}_p}^\times$ and for $p \neq \infty$, let $H_p = U_p$. Then the restricted direct product is called *ideles* and is denoted by $\mathbb{J}$. Again $\mathbb{Q}^\times$ can be embedded into $\mathbb{J}$ via the map $q \mapsto (q, q, ...) \in \mathbb{J}$ since if $q = \dfrac{a}{b} \in \mathbb{Q}, x \in U_p$ for $p \not| \, ab$.

**Proposition .** $\mathbb{Q}^\times \subset \mathbb{J}$ is discrete.

***Proof:*** Similar to the proof of the assertion that $\mathbb{Q} \subset \mathbb{A}$ is discrete, if we show that there exists a neighborhood of $1 \in \mathbb{Q}^\times$ which contains no other rational number, then we will be done. It is obvious that $(1/2, 3/2) \times \prod_p U_p$ is such a neighborhood. $\square$

In this case, however, we do not have $\mathbb{J}/\mathbb{Q}^\times$ is compact. Instead we will show that it is the direct product of $\mathbb{R}^+$ and of a compact group.

Let us define the *idelic norm* $\| \cdot \| : \mathbb{J} \to \mathbb{R}^+$ by $\|(x_p)_p\| = \prod_p |x_p|_p$. $\| \cdot \|$ is a continuous homomorphism. Let $\mathbb{J}^1$ be the kernel of $\| \cdot \|$. $\mathbb{J}^1$ is closed and $\mathbb{Q}^\times \subset \mathbb{J}^1$. Also $\mathbb{J} = \mathbb{R}^+ \times \mathbb{J}^1$. This is so because, with the identification $r \leftrightarrow (x_p)_p, x_\infty = r, x_p = 1, p$ prime, $\mathbb{J} = \mathbb{R}^+ \cdot \mathbb{J}^1$ and $\mathbb{J}^1 \cap \mathbb{R}^+ = \{1\}$.

**Proposition .** $\mathbb{J}^1/\mathbb{Q}^\times$ is compact.

***Proof:*** Let $E = \prod_p U_p$. We will show that $\mathbb{J}^1 = \mathbb{Q}^\times \cdot E$. Let $(x_p)_p \in \mathbb{J}^1$. Then $x_p \in \mathbb{Z}_p$ for $p \notin S$, $S$ a finite set of $p$'s. For $p \in S, x_p = p^{k_p} u_p$, where $u_p \in U_p, k_p \in \mathbb{Z} - \{0\}$. Let

$$r = \prod_{p \in S} p^{-k_p} \operatorname{sgn}(x_\infty) \ .$$

Then $x_p r \in U_p$ for all $p$ and $|x_\infty r| = |x_\infty \prod_{p \in S} p^{-k_p}| = \prod_p |x_p|_p = 1$. So $x_\infty r = 1$. So $E \cdot \mathbb{Q}^\times = \mathbb{J}^1$. Since $E$ is compact, we are done. $\square$

Now we will analyze the characters on $G$, the restricted direct product of $\{(G_p, H_p)\}_p$. If $c : G \to \mathbb{C}^\times$ is a character of $G$, then $c_p : G_p \hookrightarrow G \to \mathbb{C}^\times$ defines a character of $G_p$. Since $c$ is continuous, $c_p(H_p) = \{1\}$ for almost all $p$,i.e. $c_p$ is unramified for almost all $p$. Hence $c$ canbe written as $c = \prod_p c_p$. Conversely, if $\{c_p\}_p$ is a collection of characters on $G_p$'s for which $c_p(H_p) = \{1\}$ for almost all $p$, the homomorphism $c(x) = \prod_p c_p(x_p)$ for $x = (x_p)_p \in G$, is a character of $G$.

For the adeles, a character $\chi : \mathbb{A} \to \mathbb{C}^\times$ can be decomposed as $\chi(x) = \prod_p \chi_p(y_p x_p)$ for all $x = (x_p)_p \in \mathbb{A}$, where $y_p \in \mathbb{Z}_p$ for almost all $p$ to ensure that $\chi_p(\mathbb{Z}_p) = \{1\}$ for almost all $p$. This means $y = (y_p)_p \in \mathbb{A}$. Hence if we fix

$$\chi : \mathbb{A} \to \mathbb{C}^\times, \ \ \chi(x) = \prod_p \chi_p(x_p) \text{ for } x = (x_p)_p \ ,$$

to be the standard character of $\mathbb{A}$, then any character of $\mathbb{A}$ will be of the form $x \mapsto \chi(yx)$ for some $y \in \mathbb{A}$. So we have $\hat{\mathbb{A}} \cong \mathbb{A}$.

**Proposition .** In the identification defined above, $\mathbb{Q}^\perp = \mathbb{Q}$.

***Proof:*** Since $\prod_p \chi_p(q) = 1$ for $q \in \mathbb{Q}$, if $y \in \mathbb{Q}$, $\chi(yx) = 1$ for all $x \in \mathbb{Q}$. Hence $\mathbb{Q} \subset \mathbb{Q}^\perp$. We have $\mathbb{A}/\mathbb{Q}$ is compact, so $\mathbb{Q}^\perp \cong \widehat{\mathbb{A}/\mathbb{Q}}$ is discrete, hence $\mathbb{Q}^\perp/\mathbb{Q} \subset \mathbb{A}/\mathbb{Q}$ is discrete and compact, i.e. it is finite. If $n$ is the order of $\mathbb{Q}^\perp/\mathbb{Q}$, then for $y \in \mathbb{Q}^\perp \subset \mathbb{A}$, $ny \in \mathbb{Q}$, so $y \in n^{-1}\mathbb{Q} = \mathbb{Q}$. $\square$

Similarly, for the ideles, any character can be written as $c = \prod_p c_p$ with $c_p(U_p) = \{1\}$ for almost all $p$. If $p^{n_p}$ is the conductor of $c_p$, then the *conductor* of $c$ is defined to be $\prod_{p<\infty} p^{n_p}$. We will be interested in mostly the *größencharacters*, the characters which are trivial on $\mathbb{Q}^\times$. Since $\mathbb{J}/\mathbb{Q}^\times \cong \mathbb{R}^+ \times (\mathbb{J}^1/\mathbb{Q}^\times)$, any größencharacter of $\mathbb{J}$ can be written in the form $c(x) = \|x\|^s c'(x)$, where $c'$ is a character induced from $\mathbb{J}^1/{\mathbb{Q}_p}^\times$, i.e. $c'(\mathbb{R}^+) = \{1\}$ and $s = it, t \in \mathbb{R}$.

Now we will define integration on $G$, the restricted direct product of $\{(G_p, H_p)\}_p$. Since $G$ is locally compact, it has a Haar measure. If we choose Haar measures $\mu_p$ on $G_p$'s such that $\mu_p(H_p) = 1$ for almost all $p$, the Haar measure on $G$ will be fixed as follows: for a basis element $V = \prod_p V_p$ of $G$, $\mu(V) = \prod_p \mu_p(V_p)$.

**Proposition .** Suppose $f = \prod f_p$ with $f_p \in L^1(G_p)$ for all $p$, $f_p(H_p) = \{1\}$ for almost all $p$ and $\prod_p \int_{G_p} |f_p(x_p)|\, dx_p < \infty$. Then $f \in L^1(G)$ and

$$\int_G f(x)\, dx = \prod_p \int_{G_p} f_p(x_p)\, dx_p \ .$$

A *Schwartz function* on $\mathbb{A}$ is defined as a linear combination of functions of the form $f(x) = \prod_p f_p(x_p)$ where $f_p$'s are Scwartz functions on $\mathbb{Q}_p$'s and $f_p = \mathrm{char}(\mathbb{Z}_p)$ for almost all $p$. In particular for a Schwartz function, if $f = \prod f_p$, then $\hat{f} = \prod \hat{f}_p$.

## 7. Global Zeta Functions

Let $f$ be a Schwartz function on $\mathbb{A}$, $c$ a größencharacter. A *global zeta function* is a function of a complex variable $s$ defined by

$$\zeta(f, c, s) = \int_{\mathbb{J}} f(x) c(x) \|x\|^s \, d^\times x \ .$$

This integral converges absolutely and uniformly for $\mathrm{Re}\, s > 1$.
**Example**

Let $f_\infty(x) = e^{-\pi i x^2}$ on $\mathbb{R}$ and $f_p = \mathrm{char}(\mathbb{Z}_p)$ on $\mathbb{Q}_p$, and $c = \mathbf{1}$. Then

$$\begin{aligned} \zeta(f, c, s) &= \prod_p \int_{{\mathbb{Q}_p}^\times} f_p(x) |x|_p^s \, d^\times x = \prod_p \zeta(f_p, 1, s) \\ &= \pi^{1\frac{s}{2}} \Gamma(\frac{s}{2}) \prod (1 - p^{-s})^{-1} = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) \ , \end{aligned}$$

as long as the convergence of all the integrals is guaranteed, and that is the case when $\mathrm{Re}\, s > 1$.

**Theorem .** *$\zeta(f, c, s)$ possesses an analytic continuation as a meromorphic function of $s$ and satisfies the functional equation*

$$\zeta(f, c, s) = \zeta(\hat{f}, c^{-1}, 1 - s) \ .$$

***Proof:*** Recall that $\mathbb{J} = \mathbb{R}^+{\mathbb{Q}_p}^\times \prod_{p<\infty} U_p$. Let $E = \prod_{p<\infty} U_p$.

$$\begin{aligned}
\zeta(f,c,s) &= \int_{\mathbb{J}} f(x)c(x)\|x\|^s \, d^\times x \\
&= \int_0^\infty \int_{\mathbb{J}^1} f(yt)c(yt)\|yt\|^s \, d^\times y d^\times t \\
&= \int_0^\infty \left( \int_{\mathbb{J}^1} f(yt)c(yt) \, d^\times y \right) t^s \, d^\times t \ .
\end{aligned}$$

Consider now the integral inside the parantheses:

$$(\dagger) \qquad \int_{\mathbb{J}^1} f(yt)c(yt) \, d^\times y = \int_E \sum_{r\in\mathbb{Q}^\times} f(ryt)c(ryt) \, d^\times y = \int_E \sum_{r\in\mathbb{Q}^\times} f(ryt) \, c(yt) \, d^\times \ .$$

By Poisson summation formula (which will be proved in the next section) we have

$$\begin{aligned}
\sum_{r\in\mathbb{Q}^\times} f(ryt) &= \sum_{r\in\mathbb{Q}} f(ryt) \ - \ f(0) \\
&= \frac{1}{\|yt\|} \sum_{r\in\mathbb{Q}} \hat{f}(\frac{r}{yt}) \ - \ f(0) \\
&= t^{-1} \sum_{r\in\mathbb{Q}^\times} \hat{f}(\frac{r}{yt}) \ + \ t^{-1}\hat{f}(0) - f(0) \ .
\end{aligned}$$

Without loss of generality we may assume that $c$ is induced from $\mathbb{J}^1$, the equation $\dagger$ becomes

$$\begin{aligned}
\int_{\mathbb{J}^1} f(yt)c(yt) \, d^\times y &= \int_K \sum_{r\in\mathbb{Q}^\times} \hat{f}(\frac{r}{yt})c(yt)t^{-1} \, d^\times y + (t^{-1}\hat{f}(0) - f(0)) \int_K c(y) d^\times y \\
&= t^{-1} \int_{\mathbb{J}^1} \hat{f}(\frac{1}{yt})c^{-1}(\frac{1}{yt}) \, d^\times y + (t^{-1}\hat{f}(0) - f(0)) \int_K c(y) d^\times y \\
&= t^{-1} \int_{\mathbb{J}^1} \hat{f}(\frac{y}{t})c^{-1}(\frac{y}{t}) \, d^\times y + (t^{-1}\hat{f}(0) - f(0)) \times \begin{cases} 1 & \text{if } c|_K = \mathbf{1} \\ 0 & \text{if not.} \end{cases} \ .
\end{aligned}$$

Hence we have

$$\begin{aligned}
\zeta(f,c,s) &= \int_0^\infty \left( \int_{\mathbb{J}^1} f(yt)c(yt) \, d^\times y \right) t^s \, d^\times t \\
&= \int_0^1 \left( \int_{\mathbb{J}^1} \hat{f}(\frac{y}{t})c^{-1}(\frac{y}{t}) \, d^\times y \right) t^{s-1} \, d^\times t + \begin{cases} \int_0^1 (t^{s-1}\hat{f}(0) - t^s f(0)) \, d^\times t \\ 0 \end{cases} \\
&\quad + \int_1^\infty \left( \int_{\mathbb{J}^1} f(yt)c(yt) \, d^\times y \right) t^s \, d^\times t \\
&= \int_1^\infty \left( \int_{\mathbb{J}^1} \hat{f}(yt)c^{-1}(yt) \, d^\times y \right) t^{1-s} \, d^\times t + \begin{cases} \frac{1}{s-1}\hat{f}(0) - \frac{1}{s}f(0) \\ 0 \end{cases} \\
&\quad + \int_1^\infty \left( \int_{\mathbb{J}^1} f(yt)c(yt) \, d^\times y \right) t^s \, d^\times t
\end{aligned}$$

The first and the last terms of the last equation are entire in s and transform to each other under the change of $f \to \hat{f}, c \to c^{-1}, s \to 1-s$. Also the middle term transforms to itself under the same change of variables. So we have that $\zeta(f,c,s)$ extends to a meromorphic function if $c$ is non-trivial and to an entire function if $c = \mathbf{1}$. Also it satisfies the functional equation

$$\zeta(f,c,s) = \zeta(\hat{f}, c^{-1}, 1-s)\ . \qquad \square$$

## 8. Poisson Summation Formula

**Proposition** (Poisson Summation Formula)**.** If $f$ is a Schwartz function on $\mathbb{A}$, then

$$\sum_{r\in\mathbb{Q}} f(ar) = \frac{1}{\|a\|}\sum_{r\in\mathbb{Q}} \hat{f}(\frac{r}{a})$$

for $a \in \mathbb{J}$.

***Proof:*** We first prove the proposition for the case when $a = 1$.

Define $g(x) = \sum_{r\in\mathbb{Q}} f(x+r)$. $g$ can be shown to be continuous in the following way: Since $f$ is Schwartz on $\mathbb{A}$, $f_p = \text{char}(p^{n_p}\mathbb{Z}_p)$ for $p < \infty$. Let $K = [-1,1] \times \prod_p (p^{\min\{0,n_p\}}\mathbb{Z}_p)$. Let $x \in K, r \in \mathbb{Q}$. If $f_p(x_p + r) \neq 0$, then $|x_p + r|_p \leq p^{-n_p}$, and;

case 1: If $n_p \geq 0$, then $|x_p|_p \leq 1$ and $|x_p + r|_p \leq p^{-n_p}$ implies that $|r|_p \leq 1$, since otherwise $|x_p + r|_p =\max\{|x_p|_p, |r|_p\} > 1$.

case 2: If $n_p < 0$, then $|x_p|_p \leq p^{-n_p}$ and $|r|_p \leq\max\{|x_p|_p, |x_p + r|_p\} \leq p^{n_p}$.

Hence if $f(x + r) \neq 0$, then we have $r \in \dfrac{1}{N}\mathbb{Z}$ where $N = \prod_{n_p<0} p^{-n_p}$. So

$$\sum_{r\in\mathbb{Q}} |f(x+r)| = \sum_{-\infty}^{\infty} \leq \sum_{n=-\infty}^{\infty} \left| f_\infty(x_\infty + \frac{n}{N})\right|\ .$$

Since $f_\infty$ is a Schwartz function on $\mathbb{R}$, we have $|f_\infty(x_\infty + n/N)| \leq \dfrac{M}{(x_\infty + n/N)^2}$ for some $M$ and for all $x_\infty, n$. Since $|x_\infty| \leq 1$, we have

$$(x_\infty + \frac{n}{N})^2 \geq (\frac{|n|}{N} - |x_\infty|)^2 \geq (\frac{|n|}{N} - 1)^2 \geq \frac{1}{2}\frac{|n|^2}{N^2}$$

and hence

$$|f_\infty(x_\infty + \frac{n}{N})| \leq \frac{M'}{n^2}$$

for $|n| \geq 4N$. Hence $\sum_r |f(x + r)|$ converges uniformly to a continuous function on $K$, and hence on $\mathbb{A}$.

Now we can fix a Haar measures on $\mathbb{A}/\mathbb{Q}$ which will satisfy Fubini's Theorem. We have $D = [0,1) \times \prod_p \mathbb{Z}_p$ maps bijectively onto $\mathbb{A}/\mathbb{Q}$. Thinking of $g$ as a function on $\mathbb{A}/\mathbb{Q}$ we have

$$\hat{g}(r) = \int_{\mathbb{A}/\mathbb{Q}} g(x)\overline{\chi(rx)}\,dx = \int_{\mathbb{A}} f(x)\overline{\chi(rx)}\,dx = \hat{f}(r)\ .$$

Since $g$ is continuous and $\mathbb{A}/\mathbb{Q}$ is compact, Fourier inversion holds:

$$\mu^{-1}(D)\sum_{r\in\mathbb{Q}} \chi(rx)\hat{f}(r) = g(x) = \sum_{r\in\mathbb{Q}} f(x+r)\ .$$

Take $x = 0$. Then

$$\sum_{r\in\mathbb{Q}} \hat{f}(r) = \mu(D)\sum_{r\in\mathbb{Q}} f(r)\ .$$

Applying the same equation to $\hat{f}$ we get $\mu(D) = 1$.

To prove the general case define $h(x) = f(ax)$ for $x \in \mathbb{A}$. Then $h$ is a Schwartz function. Applying the first part to $h$ we have

$$\sum_{r\in\mathbb{Q}} h(r) = \sum_{r\in\mathbb{Q}} \hat{h}(r) .$$

Consider $\hat{h}(y)$.

$$\begin{aligned}\hat{h}(y) &= \int_{\mathbb{A}} h(x)\overline{\chi(yx)}\, dx \\ &= \int_{\mathbb{A}} f(ax)\overline{\chi(yx)}\, dx \\ &= \int_{\mathbb{A}} f(x)\overline{\chi(ya^{-1}x)}\|a\|^{-1}\, dx = \|a\|^{-1}\hat{f}(\frac{y}{a}) .\end{aligned}$$

So the equality follows. □

As a corollary, we have the classical Poisson Summation Formula:

$$\sum_{n\in\mathbb{Z}} f(an) = \frac{1}{|a|}\sum_{n\in\mathbb{Z}} \hat{f}(\frac{n}{a})$$

for any Schwartz function $f$ on $\mathbb{R}$ and $a \in \mathbb{R}^\times$. This can be obtained by taking $f_p = \mathrm{char}(\mathbb{Z}_p)$ for all $p < \infty$.

## 9. Dirichlet $L$-functions

A character $\chi : (\mathbb{Z}/k\mathbb{Z})^\times \to \mathbb{C}^\times, k \in \mathbb{Z}^+$, is called a *Dirichlet character.* $\chi$ is called *primitive* if there does not exist $k'|k$ with $\chi$ factoring through $(\mathbb{Z}/k\mathbb{Z})^\times \to (\mathbb{Z}/k'\mathbb{Z})^\times$. For a Dirichlet character, we can define a größencharacter $c : \mathbb{J} \to \mathbb{C}^\times$ as follows: We require that
•if $p \not| k$, then $p \neq \infty, c_p$ is unramified.
•if $p|k$, then $c_p(u) = \chi^{-1}(u)$ for $u \in U_p$. Here if $k = p^n m, U_p \to (\mathbb{Z}_p/k\mathbb{Z}_p)^\times \to (\mathbb{Z}/p^n\mathbb{Z})^\times \hookrightarrow (\mathbb{Z}/k\mathbb{Z})^\times$ is used.

These conditions determine $c$ on $\prod_{p<\infty} U_p$. By requiring $c(\mathbb{R}^+) = \{1\}$ and $c(\mathbb{Q}^\times) = \{1\}$, we completely define $c : \mathbb{J} \to \mathbb{C}^\times$ on $\mathbb{J} = \mathbb{R}^+\mathbb{Q}^\times \prod_{p<\infty} U_p$ as a direct product. Note that if $\chi$ is unramified, then conductor of $c$ is $k$.

**Proposition .** $c_p(p) = \chi(p)$ for $p \not| k, p \neq \infty$.

***Proof:*** Since $p \in \mathbb{Q}^\times$, we have

$$1 = c(p) = \prod_q c_q(p) .$$

Now $c_\infty(p) = 1$ since $p \in \mathbb{R}^+$ and $c_q(p) = 1$ for $q \not| kp$ since $p \in U_q$. Hence

$$c_p(p) = \left(\prod_{q|k} c_q(p)\right)^{-1} .$$

Here the product in the parantheses is exactly $\chi(p)^{-1}$. So $c_p(p) = \chi(p)$. □

Let $\chi$ be a Dirichlet character. The function

$$L(s,\chi) = \sum (n,k) = 1andn > 0\frac{\chi(n)}{n^s} = \prod_{p\not| k}(1-\chi(p)p^{-s})^{-1}$$

converges for $\operatorname{Re} s > 1$ and is called the *Dirichlet L-function* associated to $\chi$.

Assume $\chi$ is primitive. Let $c$ be the größencharacter associated to $\chi$ and consider $\zeta(f, c, s)$ with $f_p$'s as in the examples of section 5. Then

$$\begin{aligned}\zeta(f, c, s) &= \prod_p \zeta_p(f_p, c_p, s) \\ &= \prod_{p|k \text{ or } p=\infty} \zeta_p(f_p, c_p, s) \prod_{p \nmid k} (1 - \chi(p)p^{-s})^{-1} = L(s, \chi) \prod_{p|k \text{ or } p=\infty} \zeta_p(f_p, c_p, s) \ .\end{aligned}$$

Using the analytic continuation and functional equations of local and global zeta functions, we have the following result:
Let $\epsilon = 0$ if $c_\infty = \mathbf{1}$, $\epsilon = 1$ if $c_\infty = \operatorname{sgn}$, $W(\chi) = (-i)^\epsilon \prod_{p|k} \rho(c_p)$, and define

$$\xi(s, \chi) = (k/\pi)^{s/2} \Gamma(\frac{s + \epsilon}{2}) L(s, \chi) \ .$$

**Theorem .** *$\xi(s, \chi)$ is a meromorphic function with a simple pole at $s = 1$ if $\chi = \mathbf{1}$, and is an entire function if $\chi \neq \mathbf{1}$ and it satisfies the functional equation*

$$\xi(1 - s, \chi^{-1}) = W(\chi)\xi(s, \chi)$$

*with $|W(\chi)| = 1$.*

## References


[F] G. B. Folland, *A Course in Abstract Harmonic Analysis*, Studies in Advanced Mathematics, CRC Press, Boca Raton, 1995.

[Gel] S. Gelbart, *Automorphic Forms on Adele Groups*, Annals of Mathematics Studies, No. 83, Princeton University Press, New Jersey, 1975.

[Gol] L. J. Goldstein, *Analytic Number Theory*, Prentice-Hall, Inc., New Jersey, 1971.

[Gou] F. Q. Gouvêa, *p-adic Numbers: An Introduction*, Springer-Verlag, New York, 1993.

[S] J.-P. Serre, *A Course in Arithmetic*, Springer-Verlag, New York, 1973.

[T] J. T. Tate, "Fourier Analysis in Number Fields and Hecke's Zeta-functions", Thesis, Princeton, 1950, also appears in *Algebraic Number Theory*, Proceedings of the Brighton Conference, edited by J. W. S. Cassels and A. Fröhlich, Academic Press, New York, 1968.

[VVZ] V. S. Vladimirov, I. V. Volovich and E. I. Zelenov, *p-adic Analysis and Mathematical Physics*, World Scientific Publishing Co. Pte. Ltd., Singapore, 1994.



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