Combinatorial Interpretations of Generalized Central Factorial and Genocchi Numbers

Feryal Alayont
alayontf@gvsu.edu

Grand Valley State University

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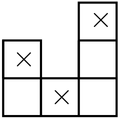
Classical Rook Theory

Example



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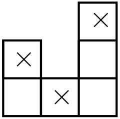
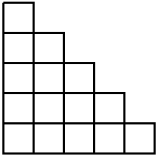


$r_k(B)$: Number of ways of placing k non-attacking rooks on B
$r_3(B) = 1, r_2(B) = 7, r_1(B) = 6, r_0(B) = 1$

Triangular boards



T_5

For size m triangular board T_m,

$$r_k(T_m) = S(m+1, m+1-k)$$

where $S(m,n)$ are the Stirling numbers of the second kind, i.e.

$$S(m,n) = S(m-1,n-1) + nS(m-1,n)$$

with $S(m,m) = 1$ and $S(m,1) = 1$.

Rooks in Three and Higher Dimensions

Question: What happens if the rooks can fly?

Follow-up: How do we want the rooks to attack in three and higher dimensions?

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Triangular Boards in Three Dimensions

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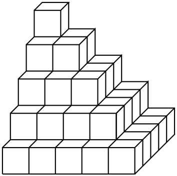


Δ_5

Theorem (Krzywonos, A.)

For size m triangle board Δ_m in three dimensions,

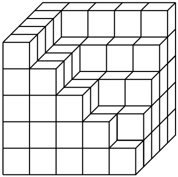
$$r_k(\Delta_m) = T(m+1, m+1-k)$$

where $T(m,n)$ are the central factorial numbers, *i.e.*

$$T(m,n) = T(m-1, n-1) + n^2 T(m-1, n)$$

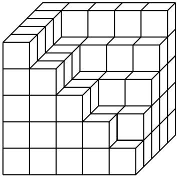
with $T(m,m) = 1$ and $T(m,1) = 1$.

Genocchi Boards in Three Dimensions



Γ_5

Genocchi Boards in Three Dimensions



Γ_5

Theorem (Krzywonos, A.)

For a size m Genocchi board Γ_m in three dimensions, $r_m(\Gamma_m)$ is given by the $(m+1)$th (unsigned even) Genocchi number $G_{2(m+1)}$ $(1, 3, 17, 155, 2073, \ldots)$

Genocchi Numbers

The generating function for the Genocchi numbers G_n is

$$\frac{2t}{e^t + 1} = \sum_{n=1}^{\infty} G_n \frac{t^n}{n!}$$

$G_{odd} = 0$ and G_{2n} count

- Permutations $a_1 a_2 \ldots a_{2n-2}$ such that even a_i is followed by a smaller number and odd a_i is followed by a larger
- Permutations $a_1 a_2 \ldots a_{2n-2}$ such that $a_{2i} < 2i$ and $a_{2i-1} \geq 2i - 1$
- Permutations $a_1 a_2 \ldots a_{2n-2}$ such that $a_i > a_{i+1}$ means both a_i and a_{i+1} are even
- Permutations $a_1 a_2 \ldots a_{2n-2}$ such that $a_i < i$ means both a_i and i are even

Rook Placements and Partitions

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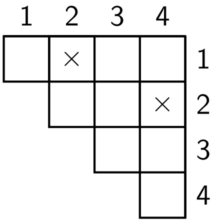
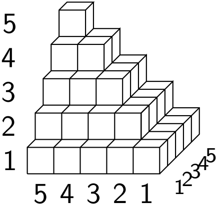
Stirling numbers of the second kind, $S(m,k)$, count partitions of m elements into k non-empty blocks.



Rook placement corresponding to partition $\{1,3\},\{2,5\},\{4\}$ of $\{1,2,3,4,5\}$

Rook Placements in 3-D and Partition Pairs



First partition: Project rooks onto the *xz*-plane
Second partition: Project onto *yz*-plane
Partition pairs (P_1, P_2) such that minimum values of the partitions are the same

Rook Placements and Restricted Permutations

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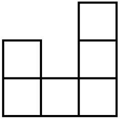
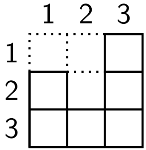
Example



Rook Placements and Restricted Permutations

Example



Rook Placements and Restricted Permutations

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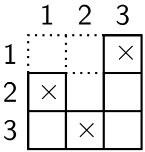

Rook Placements and Restricted Permutations

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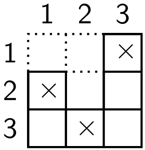

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Rook Placements in 3-D and Permutation Pairs

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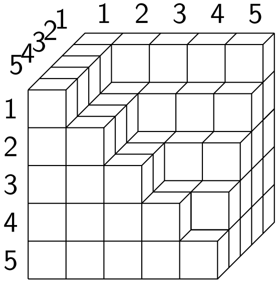
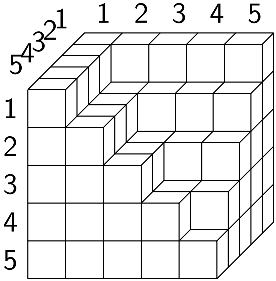


Rook Placements in 3-D and Permutation Pairs



First permutation: x coordinates of the rooks from top to bottom
Second permutation: y coordinates of the rooks from top to bottom
(π_1, π_2) where π_1, π_2 are permutations of 5 and $\pi_1(i)$ or $\pi_2(i) \leq i$ for each i.

Generalized Results in m Dimensions

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Theorem

The generalized central factorial numbers $T_d(n,k)$ count the number of ordered d-tuples $(P_1, P_2, \ldots, P_d)$ of partitions of n into k sets satisfying $\min P_1 = \min P_2 = \cdots = \min P_d$.

Theorem

Generalized (unsigned) Genocchi numbers $G_{2m}^{(d)}$ count ordered d-tuples of permutations $(\pi_1, \pi_2, \ldots, \pi_d)$ of $m-1$ such that $\min_j \pi_j(i) \leq i$ for $1 \leq i \leq m-1$.

Thanks!

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