A New Combinatorial Interpretation of Generalized Genocchi Numbers

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Overview

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- Classical rook theory, and in 3-D and beyond

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- Families of boards corresponding to Genocchi and central factorial numbers

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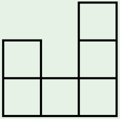

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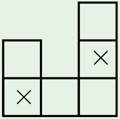

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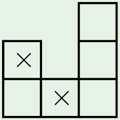

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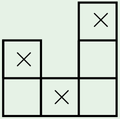

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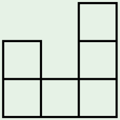

$$R_B(x) = x^3 + 7x^2 + 6x + 1$$

Triangular boards

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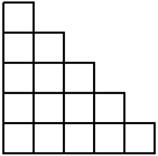
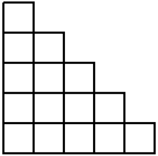


T_5

Triangular boards



For size m triangular board T_m,

$$r_k(T_m) = S(m+1, m+1-k)$$

where $S(m, n)$ are the Stirling numbers of the second kind, i.e.

$$S(m, n) = S(m-1, n-1) + nS(m-1, n)$$

with $S(m, m) = 1$ and $S(m, 1) = 1$.

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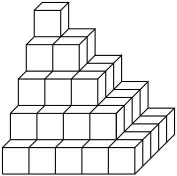
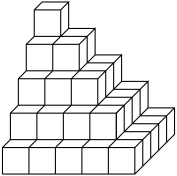


Δ_5

Triangular Boards in Three Dimensions



Δ_5

Theorem (Krzywonos, A.)

For size m triangle board Δ_m in three dimensions,

$$r_k(\Delta_m) = T(m+1, m+1-k)$$

where $T(m,n)$ are the central factorial numbers, *i.e.*

$$T(m,n) = T(m-1,n-1) + n^2 T(m-1,n)$$

with $T(m,m) = 1$ and $T(m,1) = 1$.

Genocchi Boards in Three Dimensions

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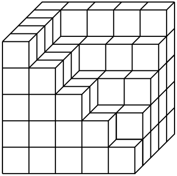
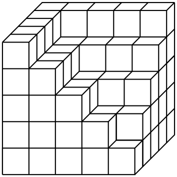


Γ_5

Genocchi Boards in Three Dimensions



Γ_5

Theorem (Krzywonos, A.)

For a size m Genocchi board Γ_m in three dimensions, $r_m(\Gamma_m)$ is given by the $(m+1)$th (unsigned even) Genocchi number $G_{2(m+1)}$ $(1, 3, 17, 155, 2073, \ldots)$

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The generating function for the Genocchi numbers G_n is

$$\frac{2t}{e^t + 1} = \sum_{n=1}^{\infty} G_n \frac{t^n}{n!}$$

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- Permutations $a_1 a_2 \ldots a_{2n-2}$ such that $a_i < i$ means both a_i and i are even

Rook Placements and Restricted Permutations

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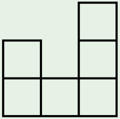

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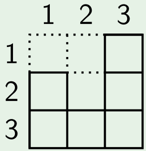

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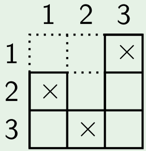

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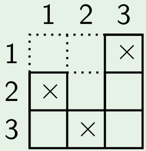

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Rook Placements in 3-D and Permutation Pairs

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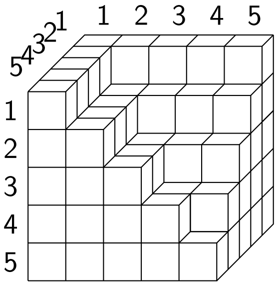

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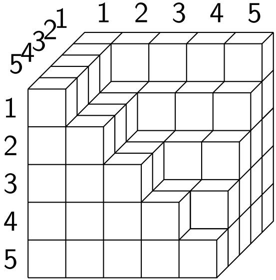

First permutation: 1st coordinates of the rooks from top to bottom
Second permutation: 2nd coordinates of the rooks from top to bottom

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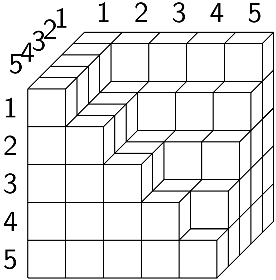

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Pairs of permutations of 5 π_1, π_2 such that $\pi_1(i)$ or $\pi_2(i) \leq i$ for each i.

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$$321, (123, 132) \longleftrightarrow G_8 = 17$$

Permutation tuples

More generally: The generalized Genocchi numbers count the number of permutation tuples such that at least one $\pi(i) \leq i$.

Thanks!

Joint work with

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