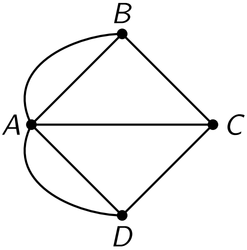
Introduction to Graph Theory

Feryal Alayont
alayontf@gvsu.edu

Grand Valley State University

December 5, 2021

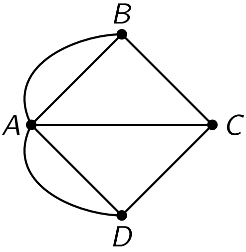
Graphs

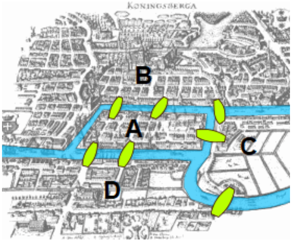


Dots (vertices -singular vertex) A, B, C, D are the objects.
Lines (edges) represent there's a relationship between the objects.
Two connected dots are said to be adjacent.

Graphs

This graph represents the city of Königsberg from the famous Bridges of Königsberg problem:

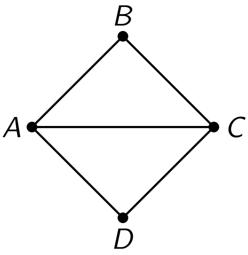




The problem asks whether it is possible to cross each bridge once (and come back to where you started).

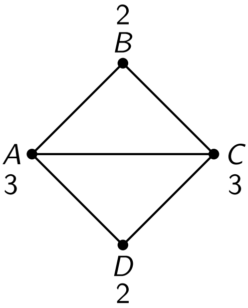
Picture from Wikipedia.

Graphs: Formally



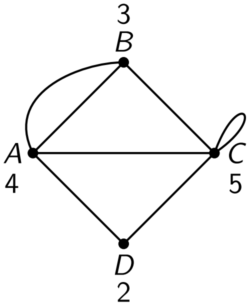
A graph formally consists of a vertex set V (in this case $\{A, B, C, D\}$) and an edge set E where each edge is a set of two vertices itself. We write edges as AB, for brevity. In this case $E = \{AB, AC, AD, BC, CD\}$.

Some definitions



Degree of a vertex: How many edges meet at that vertex; notation $\deg(v)$.

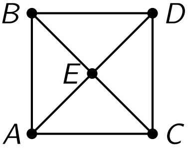
Some definitions

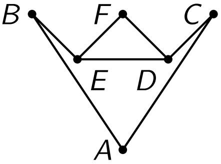


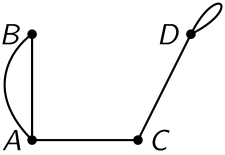
In some cases, we allow multiple connections between two vertices (multiple edges), and a connection from a vertex to itself (loop). In the multiple connections case, each edge increases the degree by 1 at each end point. In the loop case, a loop increases a degree by 2.

The relationship between degrees and edges

For each of the following graphs: **a.** Find the degrees of vertices, **b.** Find the sum of the degrees, **c.** Find the number of edges.



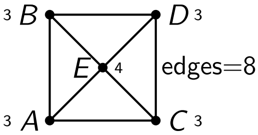


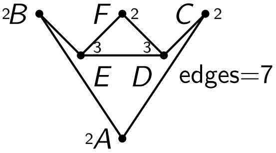


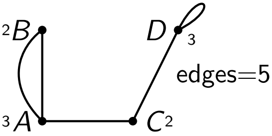
Do you notice a relationship? Can you justify?

The relationship between degrees and edges

For each of the following graphs: **a.** Find the degrees of vertices, **b.** Find the sum of the degrees, **c.** Find the number of edges.

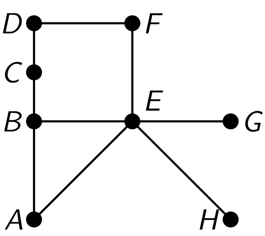


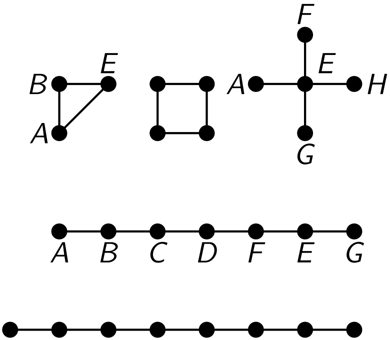




Do you notice a relationship? Can you justify?
Number of edges = Twice the sum of degrees.

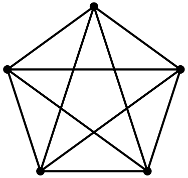
More definitions





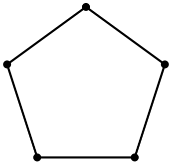
H is a *subgraph* of graph *G* if the vertices and edges of *H* are among the vertices and edges, respectively, of *G*.

Special graph families – Complete graphs



The complete graph K_n on n vertices is a graph where every vertex is connected to each of the other vertices exactly once. Shown above is K_5.

Special graph families – Cycles



The cycle C_n on n vertices is where every vertex is connected to each of its neighbors on both sides in the form of a cycle. Shown above is C_5.

Special graph families – Paths

The path P_n on n vertices is where every vertex, except for the beginning and end, is connected to each of its neighbors on both sides in the form of a line. Shown above is P_5.

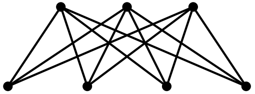
We can get a P_n from a C_n by removing an edge.

Special graph families – Paths

The path P_n on n vertices is where every vertex, except for the beginning and end, is connected to each of its neighbors on both sides in the form of a line. Shown above is P_5.

We can get a P_n from a C_n by removing an edge. Hence P_n is a subgraph of C_n.

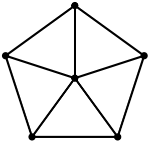
Special graph families – Complete bipartite graphs



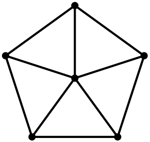
$K_{m,n}$ has two groups of vertices, m and n vertices;
every vertex in one group connects to all vertices in the other;
no connections between vertices within the same group.
Shown above is $K_{4,3}$ (same as $K_{3,4}$).

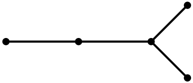
Special graphs/families – Wheels, fork, butterfly, trees

Special graphs/families – Wheels, fork, butterfly, trees

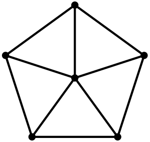
Wheel: 

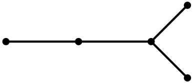
Special graphs/families – Wheels, fork, butterfly, trees

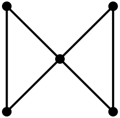
Wheel: 

Fork/chair: 

Special graphs/families – Wheels, fork, butterfly, trees

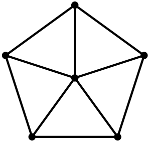
Wheel: 

Fork/chair: 

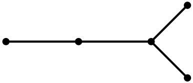
Butterfly: 

Special graphs/families – Wheels, fork, butterfly, trees

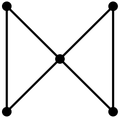
Wheel:



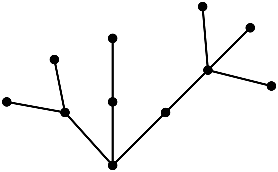
Fork/chair:



Butterfly:



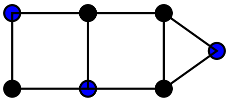
A tree:

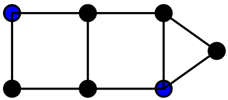


Domination

A dominating set (blue vertices) D is a subset of vertices for which any vertex not in this subset is adjacent to a vertex in D. The domination number is the number of elements in the smallest dominating set.

Domination



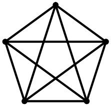


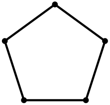
A dominating set (blue vertices) D is a subset of vertices for which any vertex not in this subset is adjacent to a vertex in D. The domination number is the number of elements in the smallest dominating set.

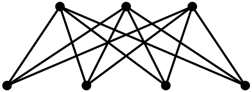
Finding the domination number is an NP-complete problem (i.e. hard algorithmically).

Domination Practice

Find the domination number of the following graphs. Some answers will depend on the graph parameter.

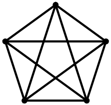
Complete K_n: 

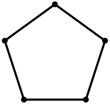
Cycle C_n: 

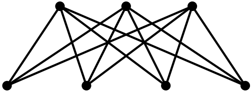
Complete bipartite $K_{m,n}$: 

Path P_n: 

Domination Practice

Complete K_n: 
Domination number=1

Cycle C_n: 
Domination number $= \lceil n/3 \rceil$

Complete bipartite $K_{m,n}$: 
Domination number=2 unless one side has only one vertex in which case it is 1.

Path P_n: 
Domination number $= \lceil n/3 \rceil$

Domination Variations

Domination Variations

k-domination: Each vertex is adjacent to at least k vertices in D.

Domination Variations

k-domination: Each vertex is adjacent to at least k vertices in D.

Connected domination: Require D to be connected.

Domination Variations

k-domination: Each vertex is adjacent to at least k vertices in D.

Connected domination: Require D to be connected.

Distance k domination: Each vertex dominates vertices at most k away.

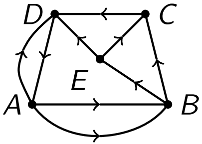
Domination Variations

k-domination: Each vertex is adjacent to at least k vertices in D.

Connected domination: Require D to be connected.

Distance k domination: Each vertex dominates vertices at most k away.

Domination in digraphs: A digraph (directed graph) is a graph where connections are directional. Think Instagram or Twitter (follow) vs. Facebook or LinkedIn connections.



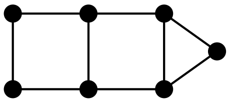
Graph labeling

Assigning labels to the vertices and/or edges of a graph satisfying certain conditions.

Graph labeling

Assigning labels to the vertices and/or edges of a graph satisfying certain conditions.

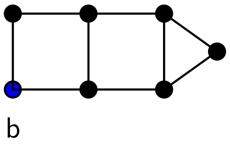
Vertex coloring: Two adjacent vertices cannot have the same color. Want minimum number of colors, called the chromatic number of the graph.



Graph labeling

Assigning labels to the vertices and/or edges of a graph satisfying certain conditions.

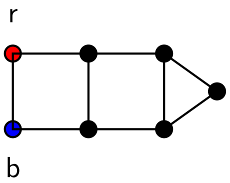
Vertex coloring: Two adjacent vertices cannot have the same color. Want minimum number of colors, called the chromatic number of the graph.



Graph labeling

Assigning labels to the vertices and/or edges of a graph satisfying certain conditions.

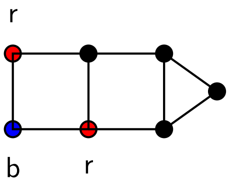
Vertex coloring: Two adjacent vertices cannot have the same color. Want minimum number of colors, called the chromatic number of the graph.



Graph labeling

Assigning labels to the vertices and/or edges of a graph satisfying certain conditions.

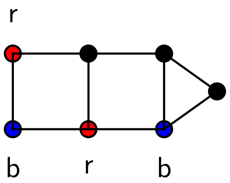
Vertex coloring: Two adjacent vertices cannot have the same color. Want minimum number of colors, called the chromatic number of the graph.



Graph labeling

Assigning labels to the vertices and/or edges of a graph satisfying certain conditions.

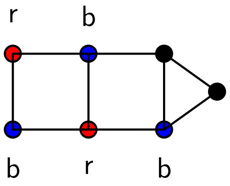
Vertex coloring: Two adjacent vertices cannot have the same color. Want minimum number of colors, called the chromatic number of the graph.



Graph labeling

Assigning labels to the vertices and/or edges of a graph satisfying certain conditions.

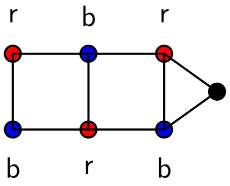
Vertex coloring: Two adjacent vertices cannot have the same color. Want minimum number of colors, called the chromatic number of the graph.



Graph labeling

Assigning labels to the vertices and/or edges of a graph satisfying certain conditions.

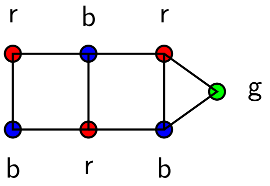
Vertex coloring: Two adjacent vertices cannot have the same color. Want minimum number of colors, called the chromatic number of the graph.



Graph labeling

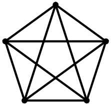
Assigning labels to the vertices and/or edges of a graph satisfying certain conditions.

Vertex coloring: Two adjacent vertices cannot have the same color. Want minimum number of colors, called the chromatic number of the graph.



Coloring Practice

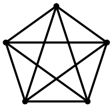
Find the chromatic number of the following graphs. Some answers will depend on the graph parameter.

Complete K_n: 

Path P_n: 

Cycle C_n: 

Coloring Practice

Complete K_n: 
Chromatic number$=n$

Path P_n: 
Chromatic number$=2$ (Paths are trees; all trees are bipartite graphs; all bipartite graphs can be colored in two colors.)

Cycle C_n: 
Chromatic number$=2$ if even, 3 if odd.

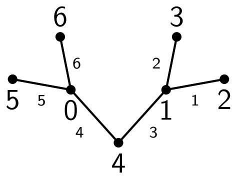
More graph labeling and variations

Prime labeling: Assign numbers $1 - n$ to n vertices of a graph so that no two adjacent vertices share a positive factor$\neq 1$.

More graph labeling and variations

Prime labeling: Assign numbers $1 - n$ to n vertices of a graph so that no two adjacent vertices share a positive factor$\neq 1$.

Graceful labeling: Assign numbers $0 - |E|$ to the vertices. For each edge, assign absolute value of the difference between the vertices. If each edge has a different label with labels $1 - |E|$, then the labeling is graceful.



Other big themes

- Hypergraphs: When a connection is between a subset of vertices rather than just two vertices, i.e. three vertices can make one connection.

Other big themes

- Hypergraphs: When a connection is between a subset of vertices rather than just two vertices, i.e. three vertices can make one connection.
- Forbidden subgraphs: Max number of edges for graphs not containing a specific subgraph.

Other big themes

- Hypergraphs: When a connection is between a subset of vertices rather than just two vertices, i.e. three vertices can make one connection.
- Forbidden subgraphs: Max number of edges for graphs not containing a specific subgraph.
- Extremal graph theory: Local-global: requiring certain local restrictions corresponds to global requirements.

Other big themes

- Hypergraphs: When a connection is between a subset of vertices rather than just two vertices, i.e. three vertices can make one connection.
- Forbidden subgraphs: Max number of edges for graphs not containing a specific subgraph.
- Extremal graph theory: Local-global: requiring certain local restrictions corresponds to global requirements.
- Spectral graph theory: Investigating properties of graphs via its matrices, like adjacency matrix.

Other big themes

- Hypergraphs: When a connection is between a subset of vertices rather than just two vertices, i.e. three vertices can make one connection.
- Forbidden subgraphs: Max number of edges for graphs not containing a specific subgraph.
- Extremal graph theory: Local-global: requiring certain local restrictions corresponds to global requirements.
- Spectral graph theory: Investigating properties of graphs via its matrices, like adjacency matrix.
- Random graphs

Other big themes

- Hypergraphs: When a connection is between a subset of vertices rather than just two vertices, i.e. three vertices can make one connection.
- Forbidden subgraphs: Max number of edges for graphs not containing a specific subgraph.
- Extremal graph theory: Local-global: requiring certain local restrictions corresponds to global requirements.
- Spectral graph theory: Investigating properties of graphs via its matrices, like adjacency matrix.
- Random graphs
- Graph minor theory

Check out for more graph stuff

- List of small graphs with names:

  https://www.graphclasses.org/smallgraphs.html

- Gallian's Dynamic Survey of Graph Labeling:

  https://www.combinatorics.org/ds6

- Some books/notes (not all working; old post):

  https://math.stackexchange.com/questions/144165/free-graph-theory-resources

- Check out recent AMS/MAA student talk abstracts for project ideas if in need of topics:

  https://www.maa.org/sites/default/files/pdf/mathfest/2021/StudentAbstractBook2021B.pdf

  https:

  //www.maa.org/sites/default/files/pdf/jmm/jmm2021/JMM_2021_Student_Poster_Abstracts.pdf

- Check out recent papers at undergraduate math journals:

  https://scholar.rose-hulman.edu/rhumj/

  https://pubs.lib.umn.edu/index.php/mjum/?

  https://msp.org/involve/about/journal/about.html

Thank you!

Thank you for listening and joining in on the graph calculations, if you were able to.

Email: alayontf@gvsu.edu