Anti-games on affine geometries

or, how getting bored with SET leads to interesting math

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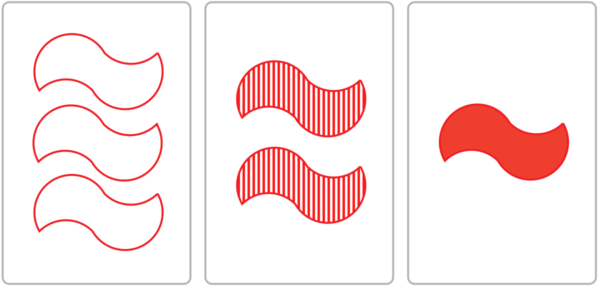
George Fisk & Nurry Goren (center)
2014 Minnesota Pi Mu Epsilon Conference

What is SET?

Color: Red, Green, Purple
Number: 1, 2, 3
Filling: Open, Stripe, Solid
Shape: Squiggle, Oval, Diamond



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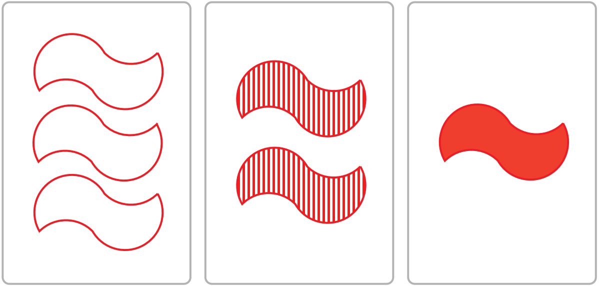

Set: 3 cards, each attribute *all same* or *all different*.

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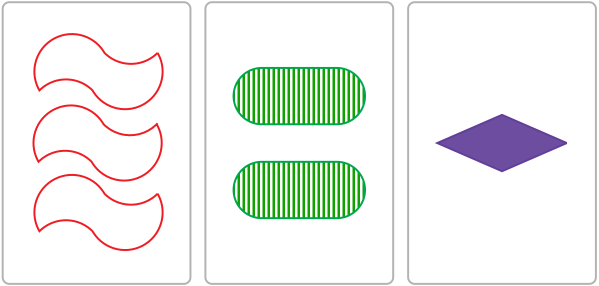

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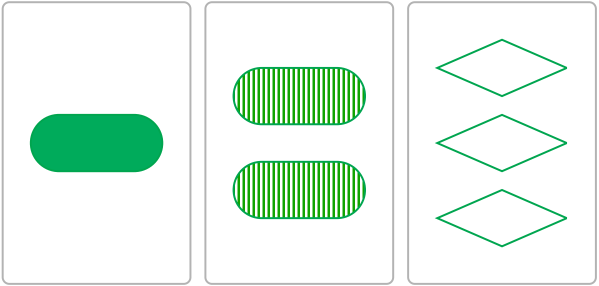

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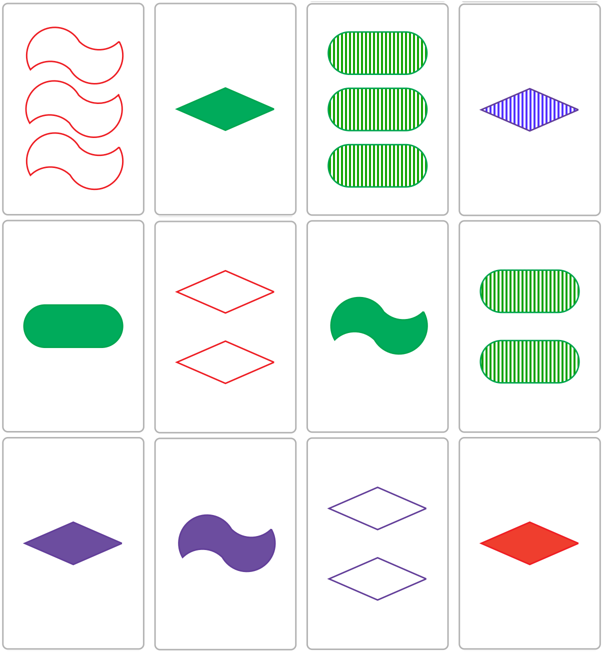

Affine Geometry

Affine Geometry $AG(n,q)$:

- **Points:** Vectors in $\mathbb{F}_q^n$
- **Lines:** 1-dim subspaces of $\mathbb{F}_q^n$ and their cosets

Affine Geometry $AG(4,3)$:

- **Points:** Vectors in $\mathbb{F}_3^4$
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$$\langle c, n, f, s \rangle$$

Affine Geometry

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$$\langle c, n, f, s\rangle$$

	Color:	Number:	Filling:	Shape:
0:	Red	3 (?!)	Open	Oval
1:	Purple	1	Stripe	Squiggle
2:	Green	2	Solid	Diamond

SET as an Affine Geometry

	c	n	f	s
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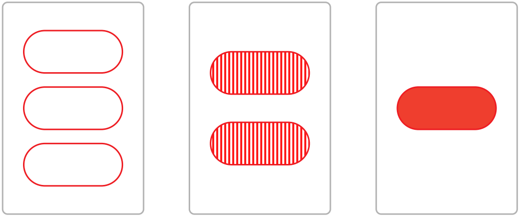
Line: 1-dim subspace of $\mathbb{F}_3^4$



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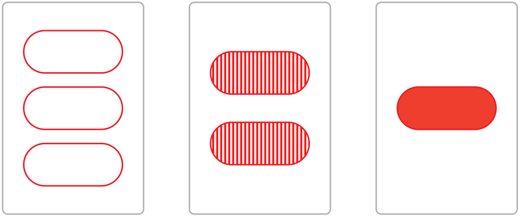
$\langle 0, 0, 0, 0 \rangle$ $\langle 0, 2, 1, 0 \rangle$ $\langle 0, 1, 2, 0 \rangle$



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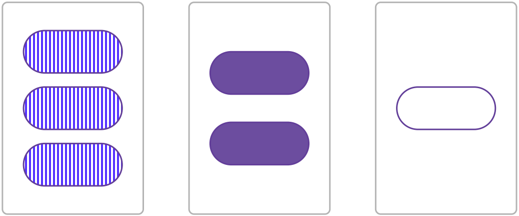

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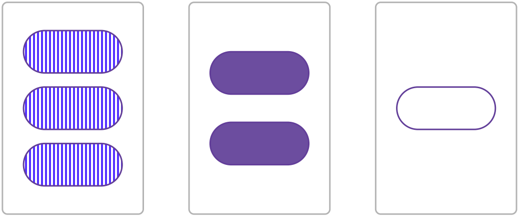
Line: 1-dim subspace of $\mathbb{F}_3^4$ **or a coset**

$$\{\langle 0,0,0,0\rangle\ \langle 0,2,1,0\rangle\ \langle 0,1,2,0\rangle\} + \langle 1,0,1,0\rangle$$



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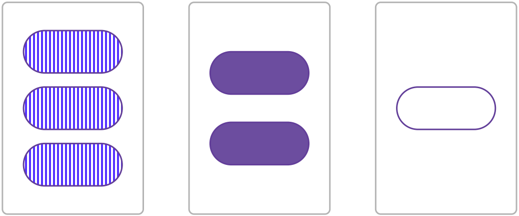
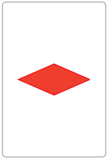
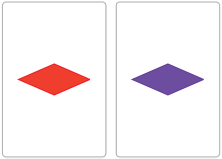
$$\{\langle 0,1,2,0\rangle\, x + \langle 1,0,1,0\rangle \,|\, x \in \mathbb{F}_3\}$$



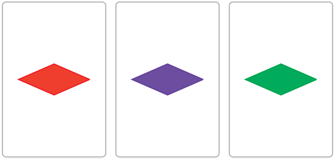
Point



Two points

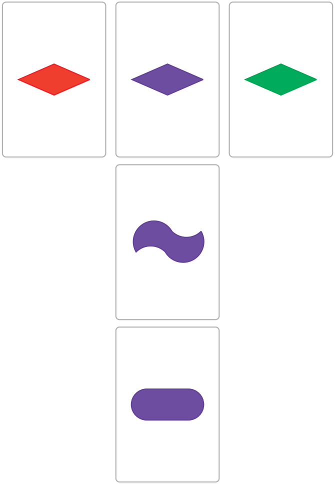


Line (set)



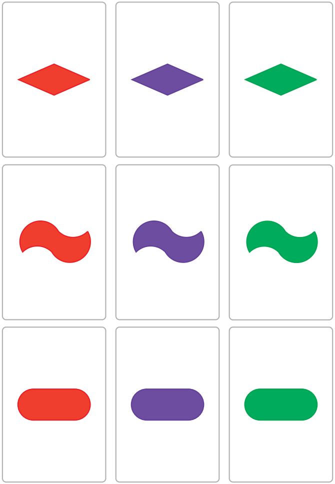
2 points determine a **unique** line

Intersecting lines



2 lines intersect in 1 point. . .

***Plane* of 3^2 cards $\cong AG(2,3)$**



…or lines can be **parallel**

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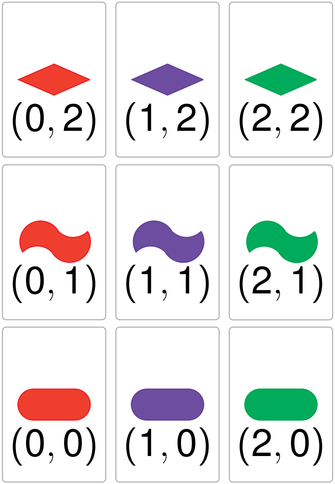
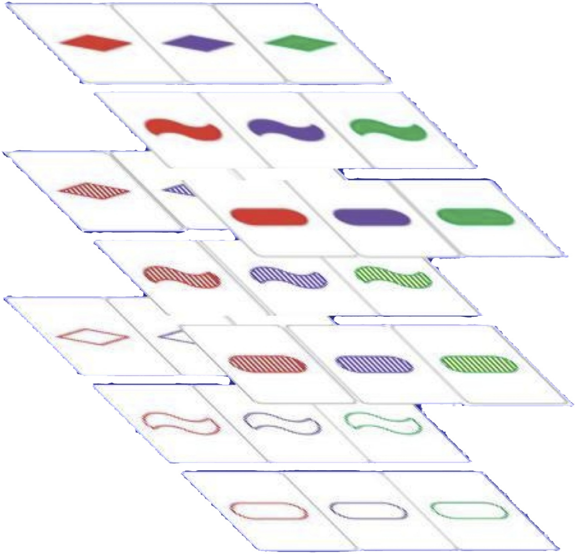
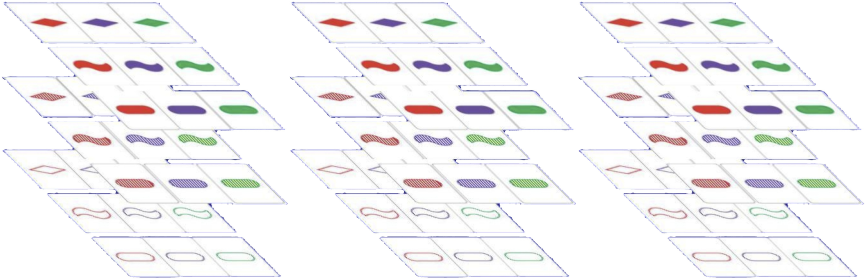


Hyperplane of 3^3 cards $\cong AG(3,3)$

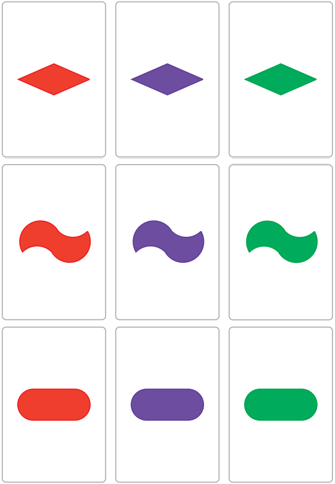


All 3^4 cards $\cong AG(4,3)$



SET as tic-tac-toe

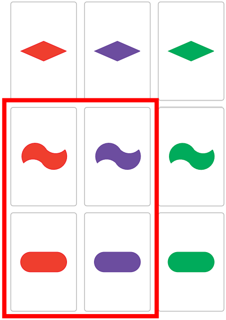
Xavier (Player 1) vs. **O**livia (Player 2)



$(0,2)$	$(1,2)$	$(2,2)$
$(0,1)$	$(1,1)$	$(2,1)$
$(0,0)$	$(1,0)$	$(2,0)$

Anti-SET

Cap in SET: A set of cards that contains no *set*.



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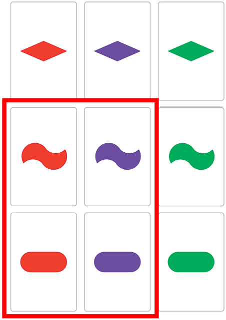

Cap in $AG(n,q)$: A set of points that contain no line.

$$\begin{array}{ccc} (0,2) & (1,2) & (2,2) \\ \\ (0,1) & (1,1) & (2,1) \\ \\ (0,0) & (1,0) & (2,0) \end{array}$$

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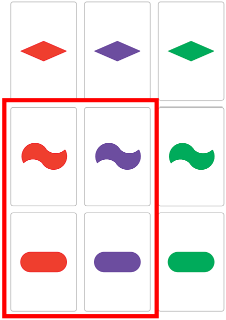

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$(0, 2)$ $(1, 2)$ $(2, 2)$

$(0, 1)$ $(1, 1)$ $(2, 1)$

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Theorem (Pellegrino, 1971):
Every set of 21 SET cards contains a *set*.

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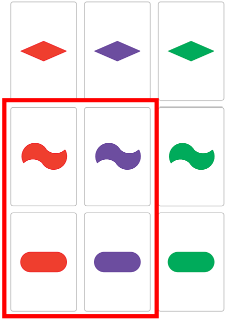

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Theorem (Pellegrino, 1971): The size of a maximal cap in $AG(4,3)$ is 20.

Anti-SET

Caps inspired **Anti-SET**:
Backwards tic-tac-toe played with SET.

- Play with all 81 SET cards
- X, O alternate taking any available card
- First to have a *set* in their hand *loses*

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Backwards tic-tac-toe played with SET.

- Play with all 81 SET cards
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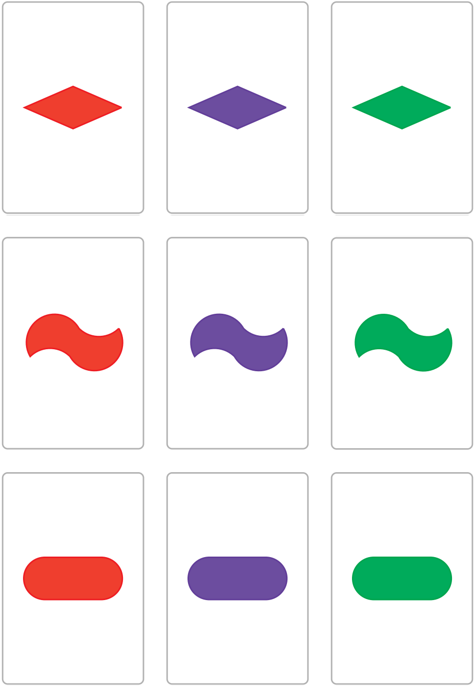
- Play with all 81 points of $AG(4,3)$.
- X, O alternate taking any available point
- First to have a line in their hand *loses*



How to win Anti-SET

Moves: $\mathcal{X}_0$, $\mathcal{O}_1$, $\mathcal{X}_1$, $\mathcal{O}_2$, $\ldots$ ($=$ points in $AG(4,3)$)

Winning Strategy for Xavier

Pick $\mathcal{X}_n$ to complete the line through $\mathcal{X}_0$ and $\mathcal{O}_n$.

Lemma: Xavier can play

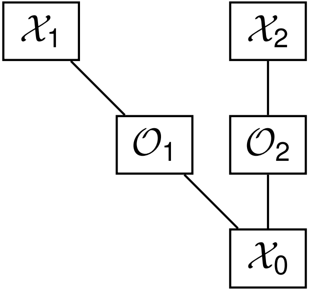
Proof: In this situation:

$$\boxed{\mathcal{X}_0} \;—\!—\!—\; \boxed{\mathcal{O}_n} \;—\!—\!—\; \boxed{??}$$

How could Xavier's move be occupied?

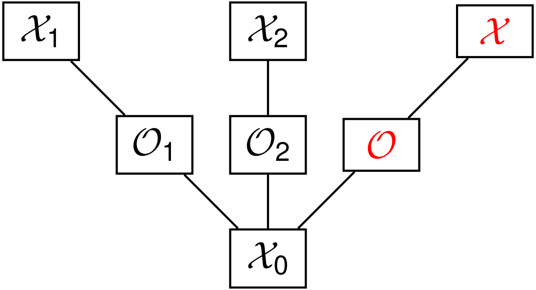
Lemma: Xavier can't lose

Proof by picture:



Vector proof

Proof by picture:



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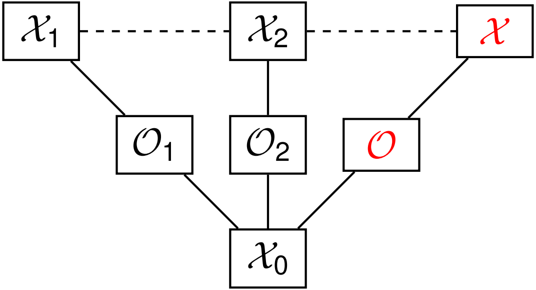

▸ Vector proof

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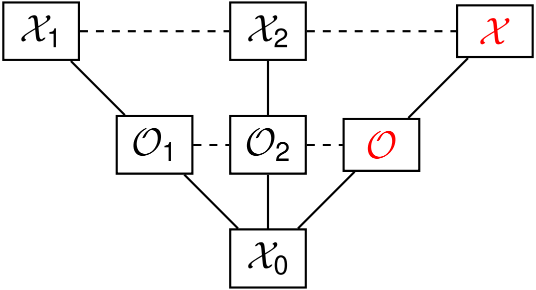
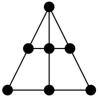


This is a *mitre* configuration: 

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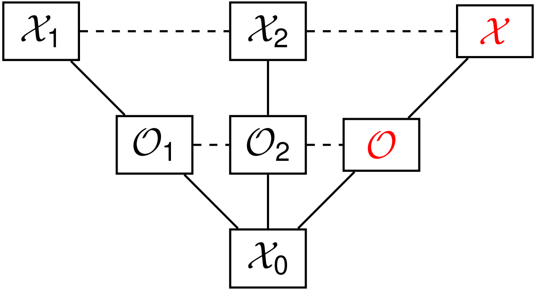
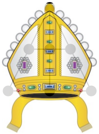


This is a *mitre* configuration: 

▸ Vector proof

Lemma: There are no ties

Proof:

- A maximal cap in $AG(4, 3)$ has size 20.
- There are 81 points.
- So, one player will eventually take a 21st card.

A similar argument works for all $AG(n, 3)$.

Detail: We don't know exact cap sizes for all $AG(n, 3)$!

Details

Summary

Theorem: Winning Strategy for Xavier

Pick $\mathcal{X}_n$ to complete the line through $\mathcal{X}_0$ and $\mathcal{O}_n$.

Proof: By the previous lemmas.

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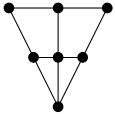
How general is this proof? In $AG(n, 3)$, $n > 1$:

- Every pair of points defines a unique line.
- There are 3 points on every line.
- There are plenty of *mitre* configurations:



How long does the game actually last?

Cap: A set of points that contains no line.

$m_2(n)$: Size of a maximal cap in $AG(n, 3)$.

$$m_2(2) = 4$$

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(Assuming Xavier follows the given strategy.)

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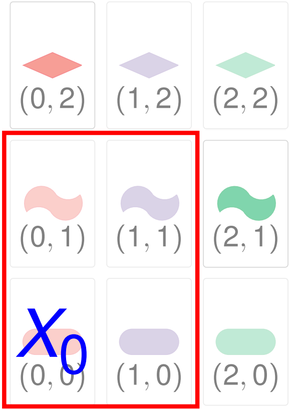
- Olivia takes every move from a maximal cap C containing X_0.



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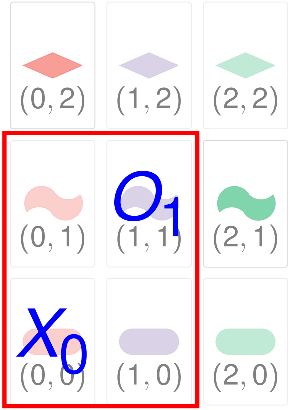
- Thus Olivia never makes a line within the cap.



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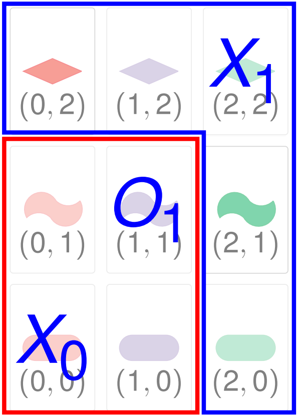
- Xavier only takes points in $\overline{C}$.



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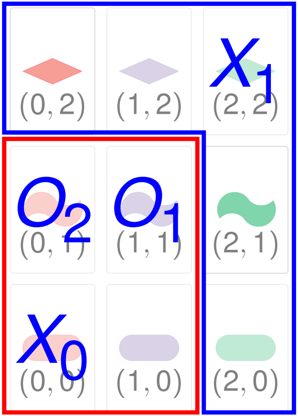

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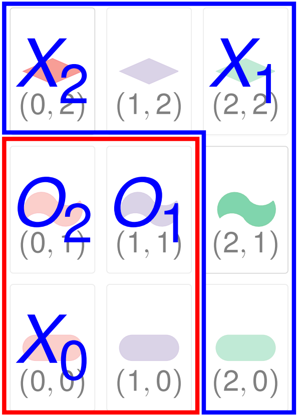

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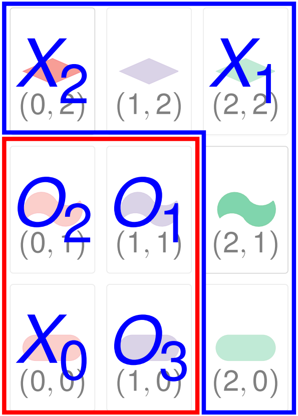

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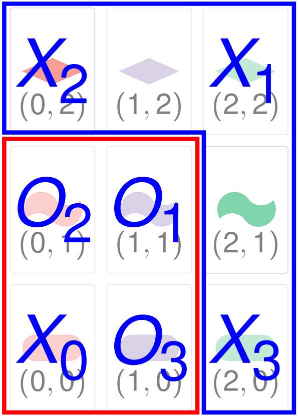

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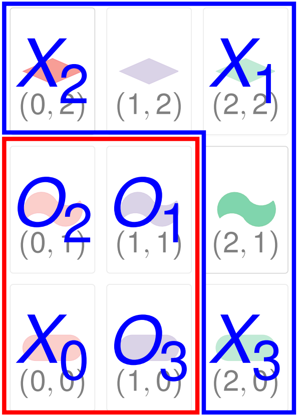
- Olivia can make one last move outside of C, guaranteed to lose.*

* Not obvious!



General Anti-Games

More options per attribute (in $AG(n, q)$, $q > 3$)

- No longer a *unique* 3rd point on each line.
- Losing condition: $\left\{\frac{q+3}{2}, \ldots, q\right\}$ points of a line?
- $\mathcal{X}$'s strategy: Could $\mathcal{O}$ steal a needed point?
  (Partial) Solution: With $X_0 = \vec{0}$, pick $X_n = -O_n$.

$$\begin{array}{c} X_2 \\ | \\ O_2 \\ | \\ X_1 \\ | \\ O_1 \\ | \\ X_0 \end{array}$$

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- Optimal play (may) depend on (k, m) *arcs:*
  k points, of which some m (but no $m + 1$) are collinear.
- Unified viewpoint: “What are the substructures from which $\mathcal{X}$ and $\mathcal{O}$ must choose their points?”

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Steiner triple systems & other Steiner designs

- Problem: Fewer mitres.

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  (Partial) Solution: With $X_0 = \vec{0}$, pick $X_n = -O_n$.
- Optimal play (may) depend on (k, m) *arcs:*
  k points, of which some m (but no $m + 1$) are collinear.
- Unified viewpoint: "What are the substructures from which $\mathcal{X}$ and $\mathcal{O}$ must choose their points?"

X_2
|
O_2
|
X_1
|
O_1
|
X_0

Steiner triple systems & other Steiner designs

- Problem: Fewer mitres.

Special cases are great projects for undergraduates!

Questions?

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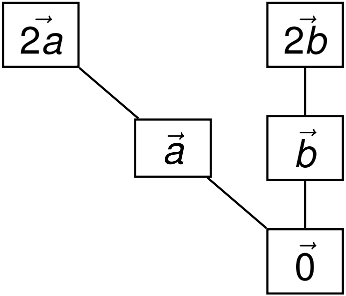
(Samuel Morse born April 27, 1791)

References

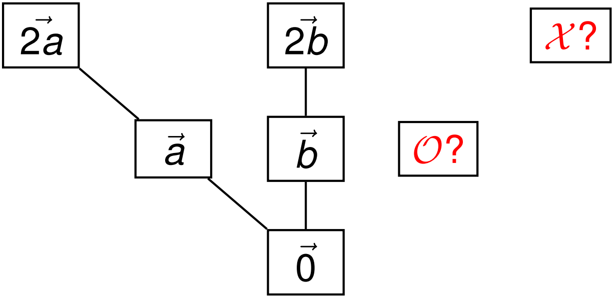
More information:

- David Clark, George Fisk, and Nurry Goren: *A variation on the game SET*.
  Involve 9 (2) (2016) 249–264.
- Benjamin Lent Davis and Diane Maclagan: *The card game SET*.
  Mathematical Intelligencer 25 (3) (2003) 33–40.
- Maureen T. Carroll and Steven T. Dougherty: *Tic-Tac-Toe on a finite plane*.
  Mathematics Magazine 77 (4) (2004) 260–274.

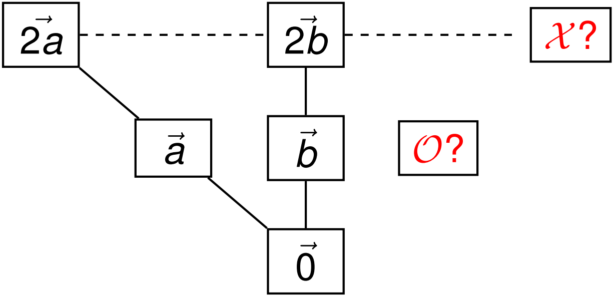
Proof: Xavier can't lose



Proof: Xavier can't lose

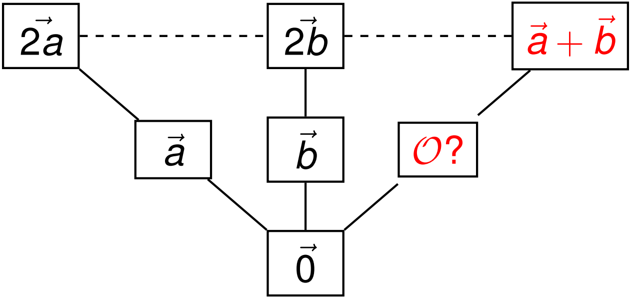


Proof: Xavier can't lose



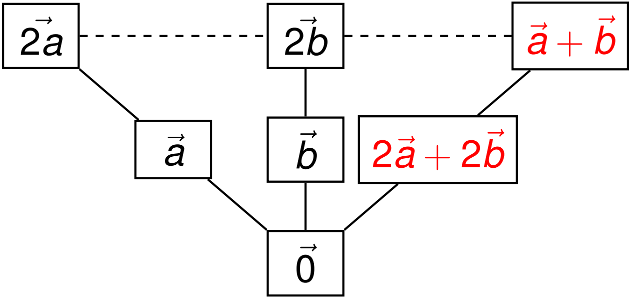
If Xavier has a line, then $\mathcal{X} = \vec{a} + \vec{b}$

Proof: Xavier can't lose



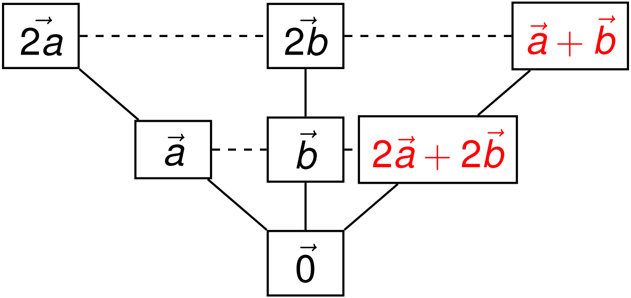
If Xavier has a line, then $\mathcal{X} = \vec{a} + \vec{b}$
By the strategy, $\mathcal{X} + \mathcal{O} + \vec{0} = \vec{0}$

Proof: Xavier can't lose



If Xavier has a line, then $\mathcal{X} = \vec{a} + \vec{b}$
By the strategy, $\mathcal{X} + \mathcal{O} + \vec{0} = \vec{0}$
So, $\mathcal{O} = 2\vec{a} + 2\vec{b}$

Proof: Xavier can't lose



If Xavier has a line, then $\mathcal{X} = \vec{a} + \vec{b}$

By the strategy, $\mathcal{X} + \mathcal{O} + \vec{0} = \vec{0}$

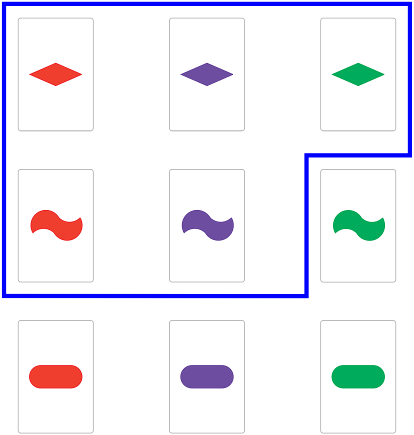
So, $\mathcal{O} = 2\vec{a} + 2\vec{b}$

$\vec{a} + \vec{b} + \mathcal{O} = \vec{0}$, so Ophelia chose a line first.

Back

Proof: There are no ties

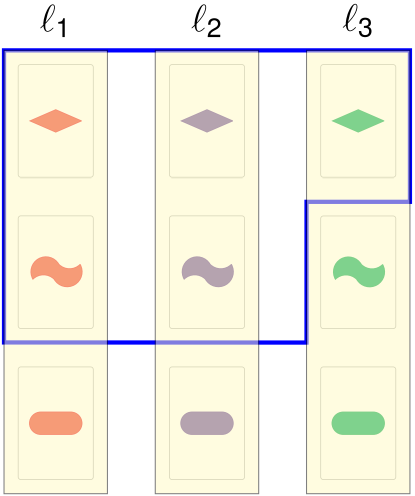
Choose a set S of 5 points:



Back

Proof: There are no ties

Choose 3 parallel lines not contained in S.



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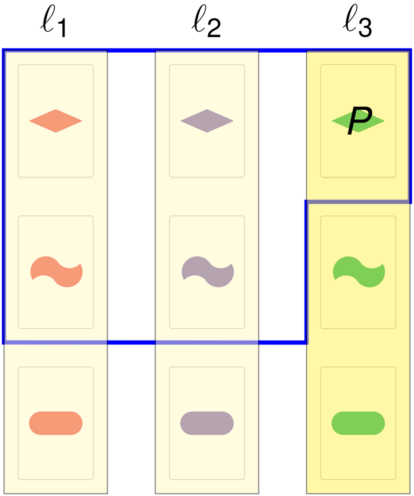
Proof: There are no ties

One of these meets S in 1 point P.



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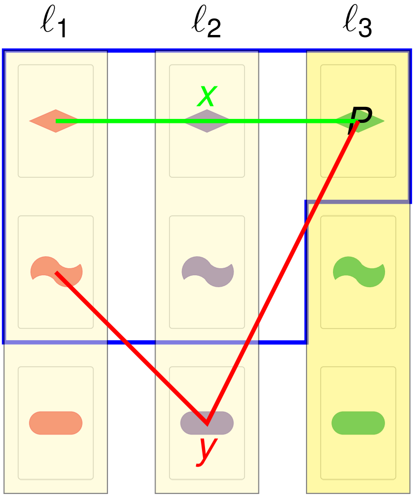
Proof: There are no ties

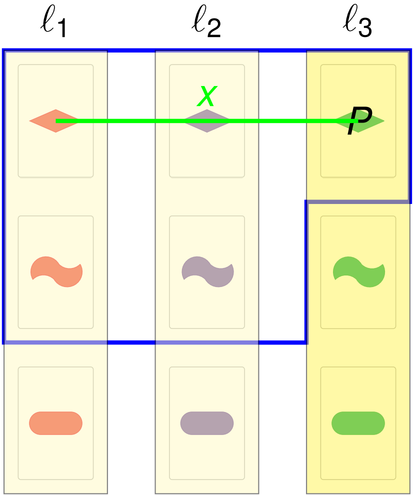
Draw lines connecting P to the two points in $\ell_1 \cap S$



Back

Proof: There are no ties

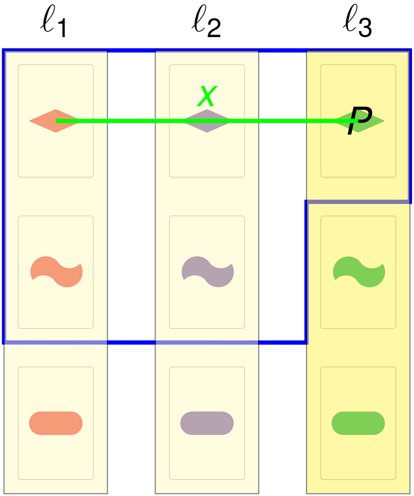
Each line meets ℓ_2 at a different point. But ℓ_2 has only 1 point not in S. So one of those points must be in S.



Back

Proof: There are no ties

This line is contained entirely in S.



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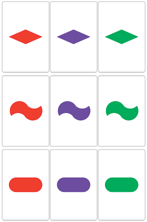
Lemma: There are no ties

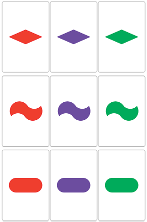
Proof:

- Play two full turns.
  This gives 2 lines which span a 9-point plane P.



Proof:

- Play two full turns.
  This gives 2 lines which span a 9-point plane P.

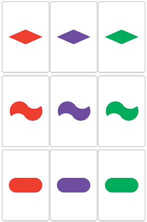


- There is no tie in P:
  Maximal cap in $AG(2,3)$ has size 4. ▸ Proof

Lemma: There are no ties

Proof:

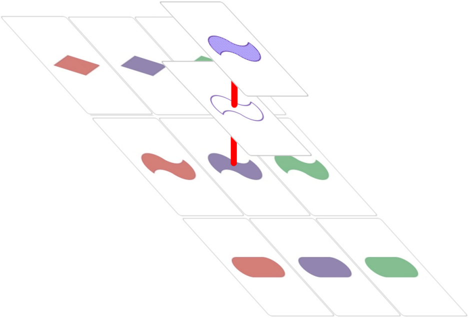
- Play two full turns.
  This gives 2 lines which span a 9-point plane P.



- There is no tie in P:
  Maximal cap in $AG(2,3)$ has size 4. ▸ Proof
- If Olivia keeps playing in P, she will lose.

Lemma: There are no ties

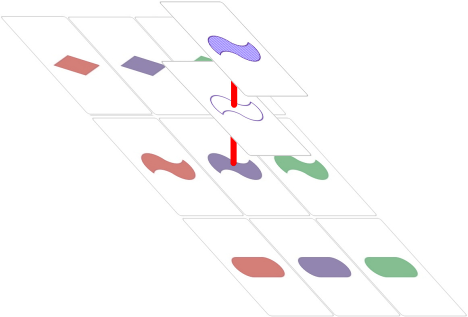
- If Olivia plays outside of P, Xavier still can't lose.
  Xavier's strategy never picks a point in P.



◂ Back

Lemma: There are no ties

- If Olivia plays outside of P, Xavier still can't lose. *Xavier's strategy never picks a point in P.*



- Therefore, Olivia must either lose outside of P, or eventually play in P again. . . and lose.

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