Anti-SET

or, how getting bored with SET leads to interesting math

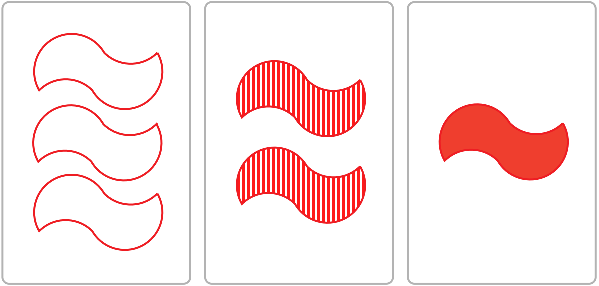
George Fisk & Nurry Goren (center)
Spring 2014 Minnesota Pi Mu Epsilon Conference

Color: Red, Green, Purple

Number: 1, 2, 3

Filling: Open, Stripe, Solid

Shape: Squiggle, Oval, Diamond



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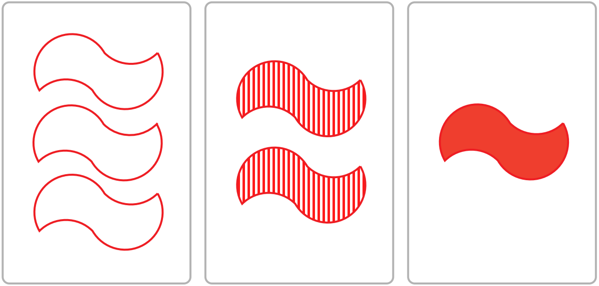

Set: 3 cards, each attribute *all same* or *all different*.

Color: Red, Green, Purple

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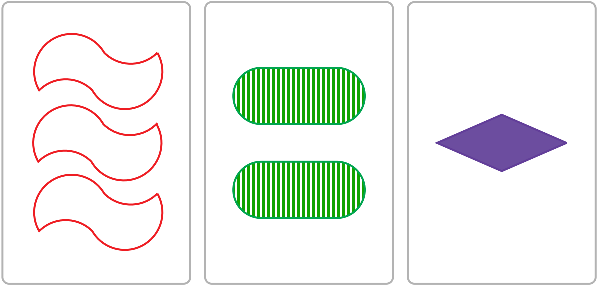

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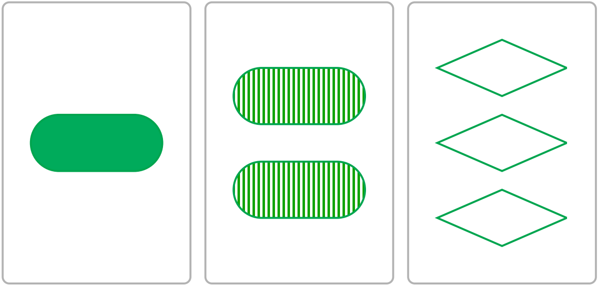
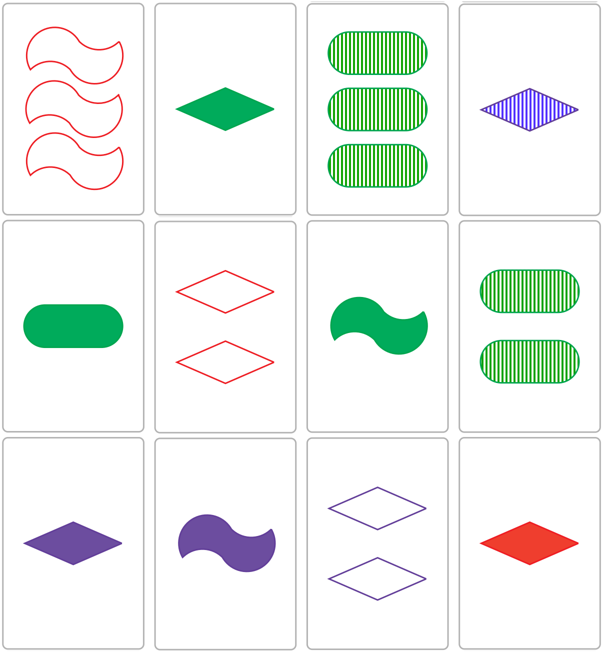


Set: 3 cards, each attribute *all same* or *all different*.

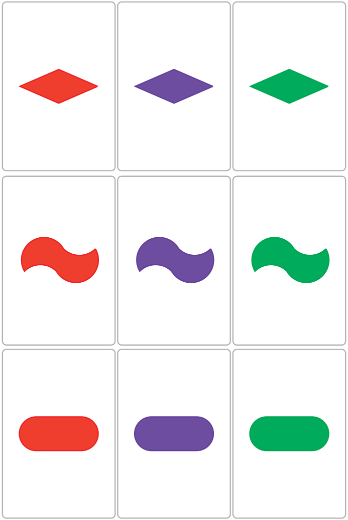


How many cards can you have without having a set?

Theorem (Pellegrino, 1971)

Every set of SET cards contains a set.

Xavier (Player 1) vs. **O**livia (Player 2)

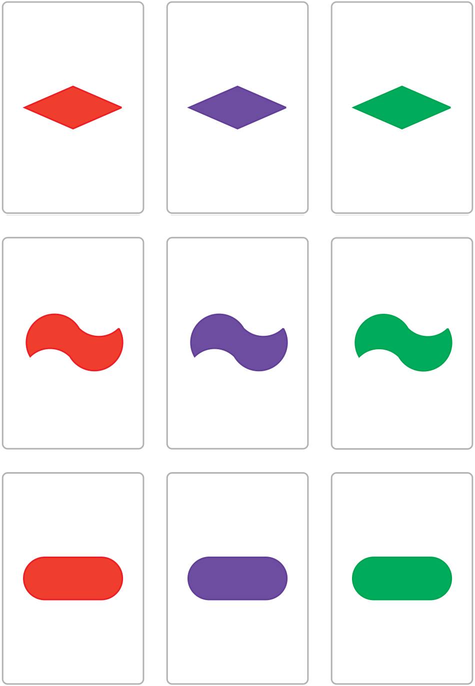


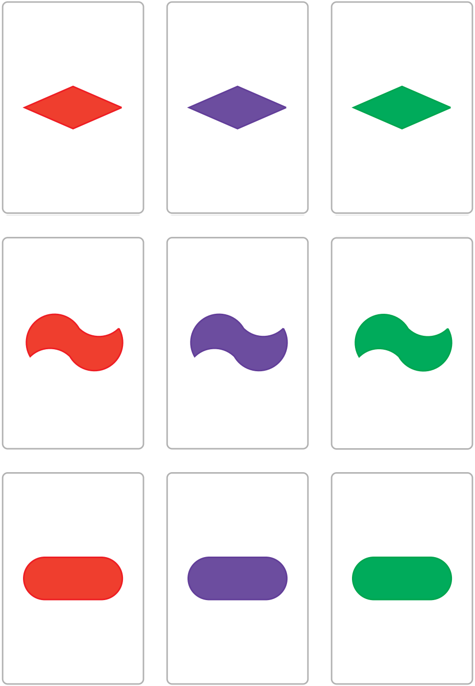
Theorem (Pellegrino, 1971)

Every set of 21 SET cards contains a set.

Anti-SET Rules

- Start with all 81 SET cards
- 2 players alternate taking any available card, tic-tac-toe style
- First to have a set in their hand *loses*

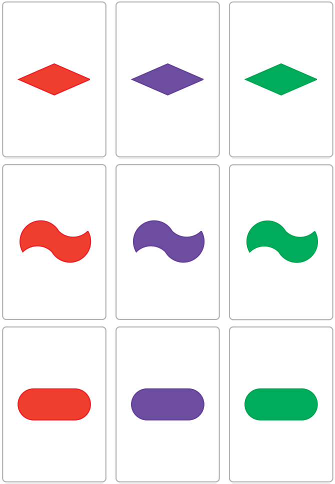




Moves: $\mathcal{X}_0, \mathcal{O}_0, \mathcal{X}_1, \mathcal{O}_1, \ldots$

Winning Strategy for Xavier

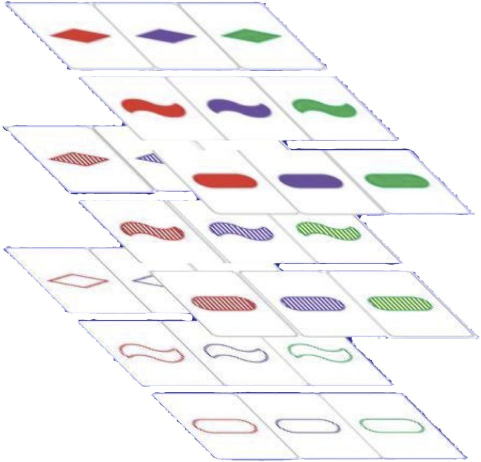
Pick $\mathcal{X}_n \ldots$



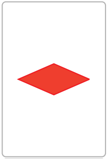
Moves: $\mathcal{X}_0$, $\mathcal{O}_0$, $\mathcal{X}_1$, $\mathcal{O}_1$, ...

Winning Strategy for Xavier

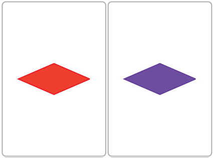
Pick $\mathcal{X}_n$ to complete the set through $\mathcal{X}_0$ and $\mathcal{O}_{n-1}$.



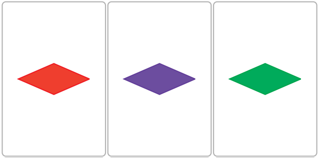
Point



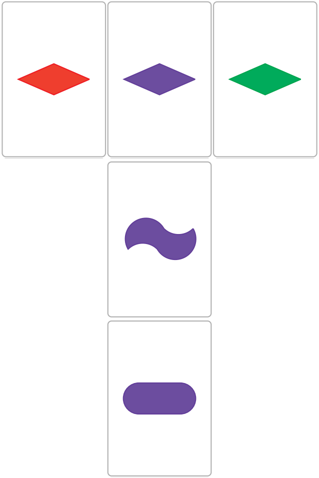
Two points form a...



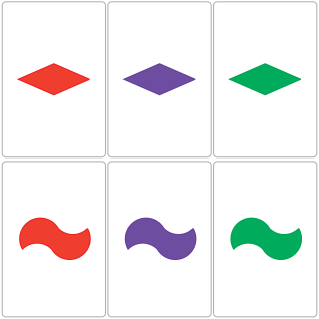
Two points form a...



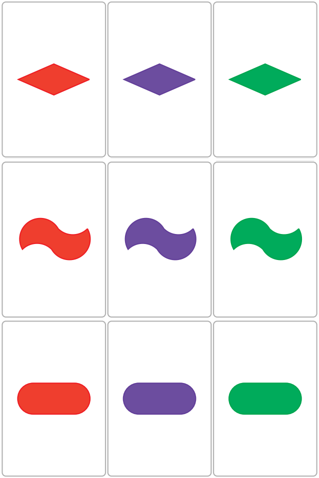
Two lines intersect in . . .



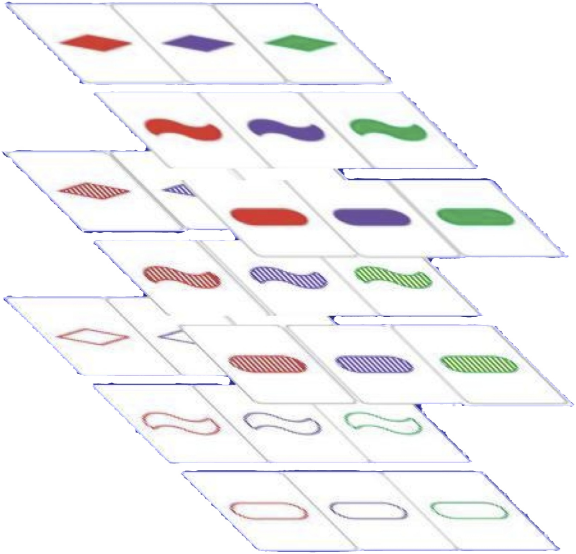
Or else they are. . .



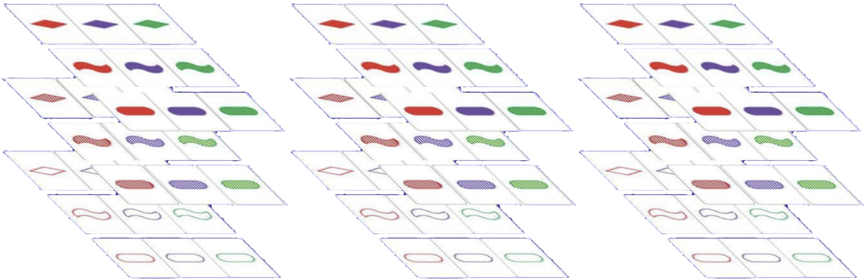
Plane **of** $3^2 = 9$ **cards**



Hyperplane of $3^3 = 27$ cards ("3D space")



All $3^4 = 81$ cards ("4D space")



Winning Strategy for Xavier

Pick $\mathcal{X}_n$ to complete the **line** through $\mathcal{X}_0$ and $\mathcal{O}_{n-1}$.

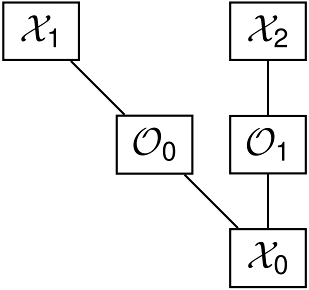
Lemma: Xavier can play.

Proof:

$$\boxed{\mathcal{X}_0} \text{ — } \boxed{\mathcal{O}_{n-1}} \text{ — } \boxed{??}$$

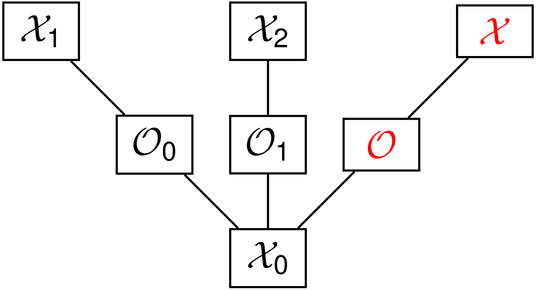
Lemma: Xavier can't lose.

Proof:



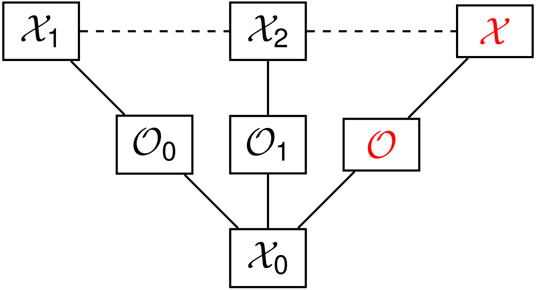
Lemma: Xavier can't lose.

Proof:



Lemma: Xavier can't lose.

Proof:



Lemma: Xavier can't lose.

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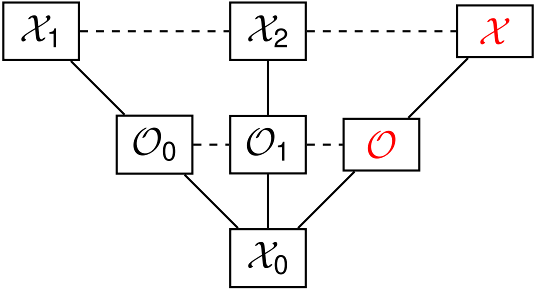
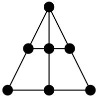


This is called a *mitre*: 

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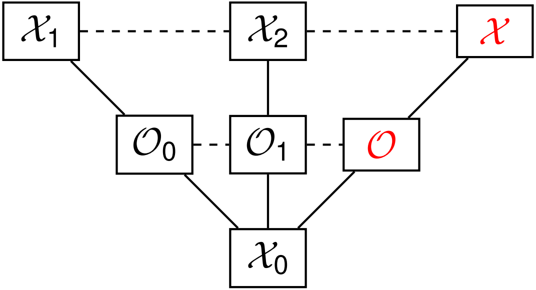
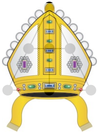
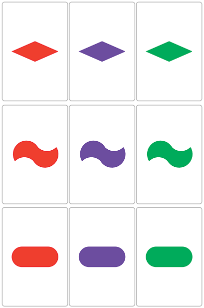
Proof:

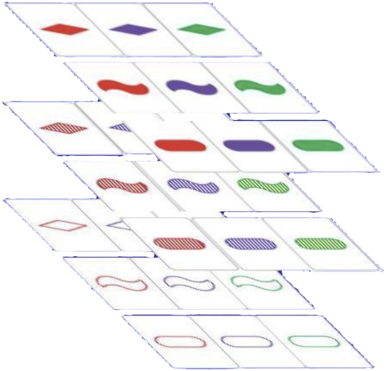


This is called a *mitre*: 

Lemma: There are no ties.

Proof:



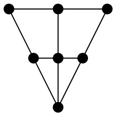


Theorem: Winning Strategy for Xavier

Pick $\mathcal{X}_n$ to complete the line through $\mathcal{X}_0$ and $\mathcal{O}_{n-1}$.

But wait... our proofs only needed:

-
-
-



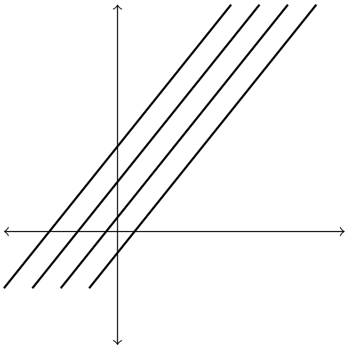
SET wasn't involved!

SET is an **affine geometry:**

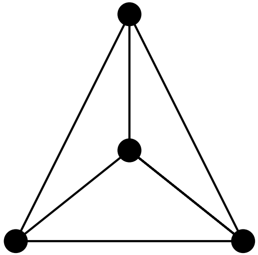
A set of “points” and “lines” such that:

- Every pair of points defines a unique line.
- Every line has the same number of points.
- Every line is part of a *parallel class* (giving mitres!).

My favorite affine geometries

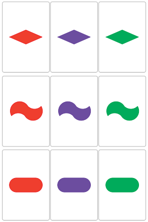


The Euclidean Plane

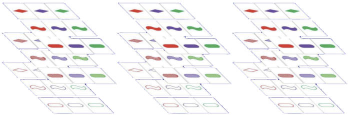


The 4-point plane

My favorite affine geometries



A 9-card SET Plane



All 81 SET Cards

We can represent SET cards as points:

$$\langle c, n, f, s \rangle$$

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$$\langle c, n, f, s \rangle$$

Color:

0: Red

1: Purple

2: Green

We can represent SET cards as points:

$$\langle c, n, f, s \rangle$$

	Color:	Number:
0:	Red	3 (?!)
1:	Purple	1
2:	Green	2

We can represent SET cards as points:

$$\langle c, n, f, s \rangle$$

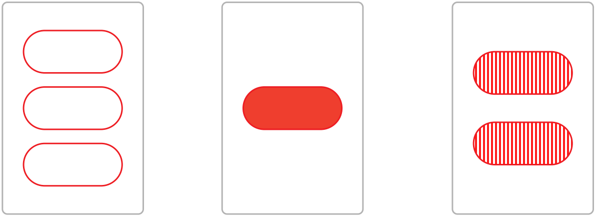
	Color:	Number:	Filling:
0:	Red	3 (?!)	Open
1:	Purple	1	Stripe
2:	Green	2	Solid

We can represent SET cards as points:

$$\langle c, n, f, s \rangle$$

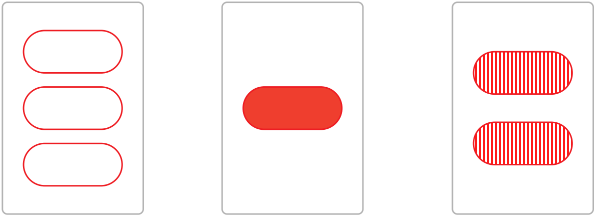
	Color:	Number:	Filling:	Shape:
0:	Red	3 (?!)	Open	Oval
1:	Purple	1	Stripe	Squiggle
2:	Green	2	Solid	Diamond

	c	n	f	s
0	Red	3	Open	Oval
1	Purple	1	Stripe	Squiggle
2	Green	2	Solid	Diamond



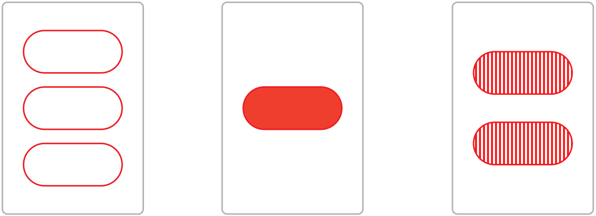
	c	n	f	s
0	Red	3	Open	Oval
1	Purple	1	Stripe	Squiggle
2	Green	2	Solid	Diamond

$\langle 0, 0, 0, 0 \rangle$



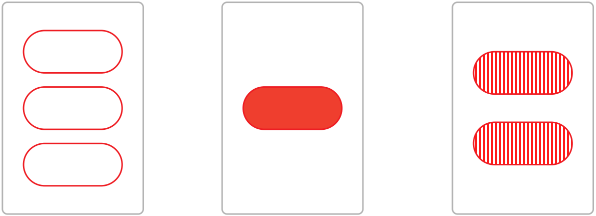
	c	n	f	s
0	Red	3	Open	Oval
1	Purple	1	Stripe	Squiggle
2	Green	2	Solid	Diamond

$\langle 0, 0, 0, 0 \rangle$ $\langle 0, 1, 2, 0 \rangle$

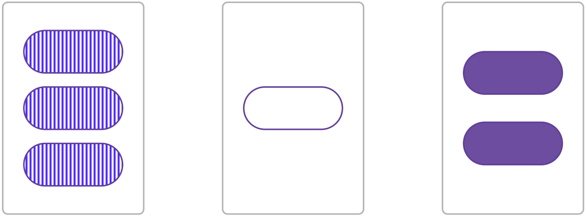


	c	**n**	**f**	**s**
0	Red	3	Open	Oval
1	Purple	1	Stripe	Squiggle
2	Green	2	Solid	Diamond

$\langle 0, 0, 0, 0 \rangle$ $\langle 0, 1, 2, 0 \rangle$ $\langle 0, 2, 1, 0 \rangle$

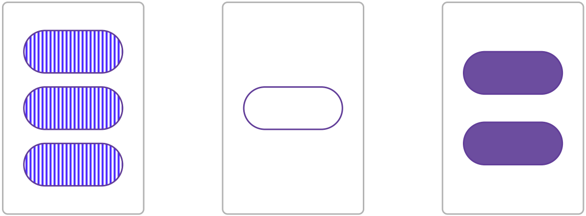


	c	**n**	**f**	**s**
0	Red	3	Open	Oval
1	Purple	1	Stripe	Squiggle
2	Green	2	Solid	Diamond



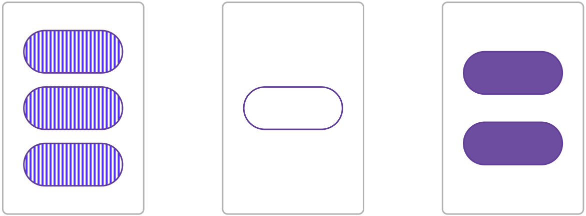
	c	n	f	s
0	Red	3	Open	Oval
1	Purple	1	Stripe	Squiggle
2	Green	2	Solid	Diamond

$\langle 1, 0, 1, 0 \rangle$



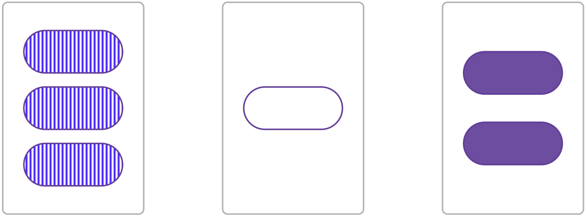
	c	n	f	s
0	Red	3	Open	Oval
1	Purple	1	Stripe	Squiggle
2	Green	2	Solid	Diamond

$\langle 1, 0, 1, 0 \rangle$ $\quad$ $\langle 1, 1, 0, 0 \rangle$

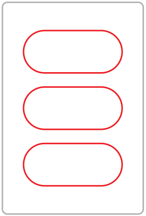


	c	n	f	s
0	Red	3	Open	Oval
1	Purple	1	Stripe	Squiggle
2	Green	2	Solid	Diamond

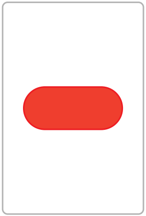
$\langle 1, 0, 1, 0 \rangle$ $\langle 1, 1, 0, 0 \rangle$ $\langle 1, 2, 2, 0 \rangle$



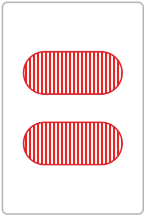
$\langle 0, 0, 0, 0 \rangle$



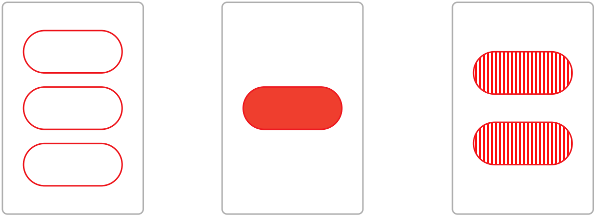
$\langle 0, 1, 2, 0 \rangle$



$\langle 0, 2, 1, 0 \rangle$



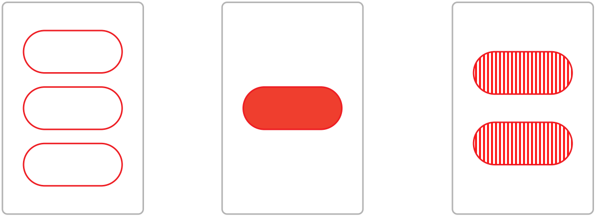
$$\langle 0,0,0,0\rangle + \langle 0,1,2,0\rangle + \langle 0,2,1,0\rangle$$



(mod 3)

$$0 \cdot \langle 0, 1, 2, 0 \rangle \quad 1 \cdot \langle 0, 1, 2, 0 \rangle \quad 2 \cdot \langle 0, 1, 2, 0 \rangle$$

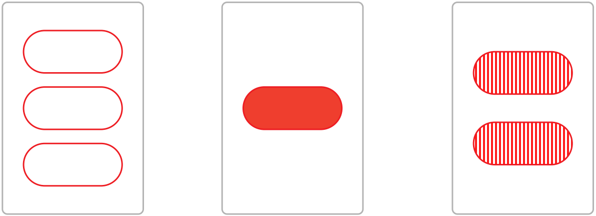
$$\langle 0, 0, 0, 0 \rangle \quad \langle 0, 1, 2, 0 \rangle \quad \langle 0, 2, 1, 0 \rangle$$



(mod 3)

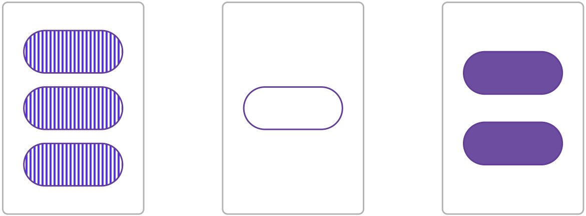
$$0 \cdot \langle 0, 1, 2, 0 \rangle \quad 1 \cdot \langle 0, 1, 2, 0 \rangle \quad 2 \cdot \langle 0, 1, 2, 0 \rangle$$

$$\langle 0, 1, 2, 0 \rangle x \pmod 3, \; x = 0, 1, 2$$



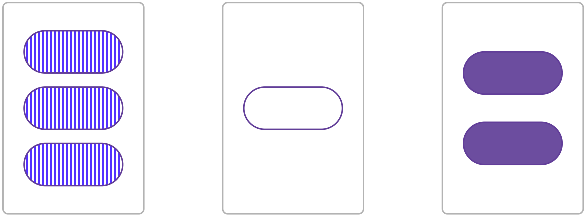
(mod 3)

$$\langle 1, 0, 1, 0 \rangle + \langle 1, 1, 0, 0 \rangle + \langle 1, 2, 2, 0 \rangle$$



(mod 3)

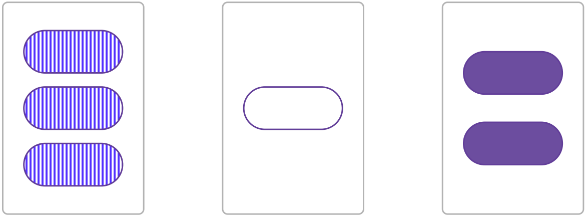
$$\begin{array}{rrr} \langle 0,0,0,0\rangle & \langle 0,1,2,0\rangle & \langle 0,2,1,0\rangle \\ +\langle 1,0,1,0\rangle & +\langle 1,0,1,0\rangle & +\langle 1,0,1,0\rangle \\ \hline \langle 1,0,1,0\rangle & \langle 1,1,0,0\rangle & \langle 1,2,2,0\rangle \end{array}$$



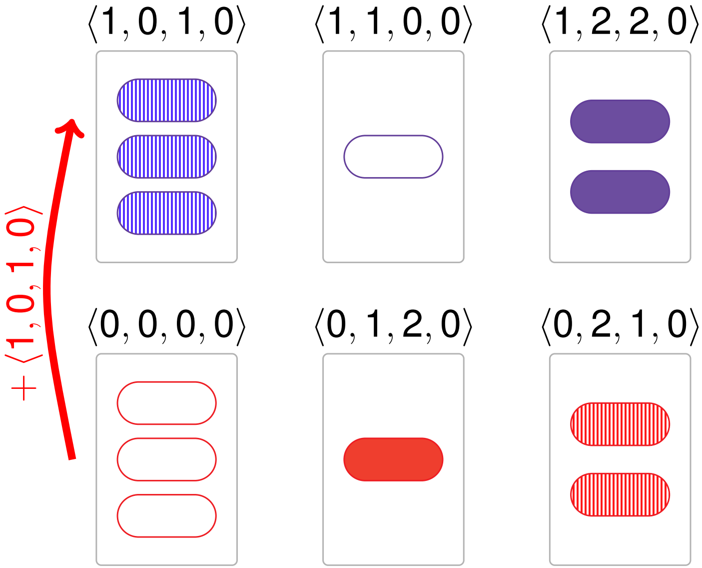
(mod 3)

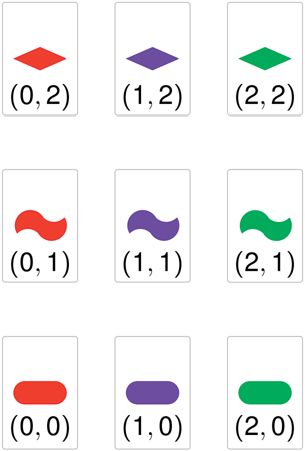
$$
\begin{array}{rrr}
\langle 0,0,0,0\rangle & \langle 0,1,2,0\rangle & \langle 0,2,1,0\rangle \\
+\langle 1,0,1,0\rangle & +\langle 1,0,1,0\rangle & +\langle 1,0,1,0\rangle \\
\hline
\end{array}
$$

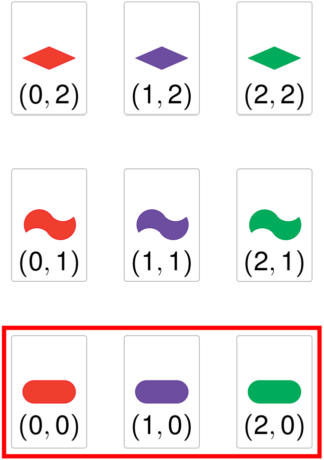
$$\langle 0,1,2,0\rangle x + \langle 1,0,1,0\rangle \ , \ x = 0,1,2$$



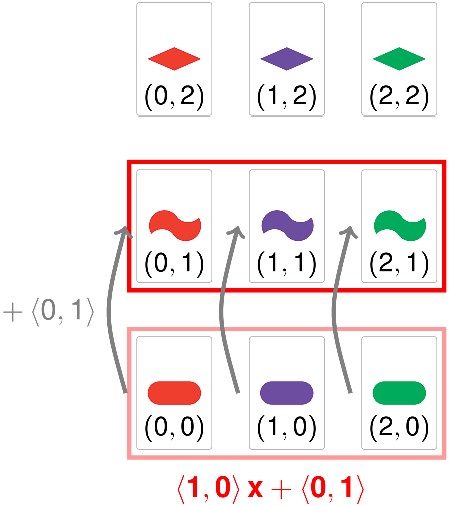
(mod 3)

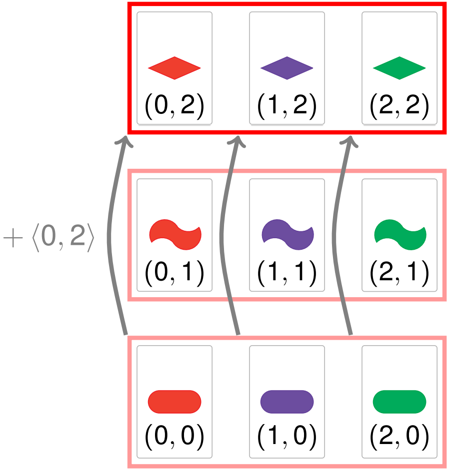

⟨1, 0, 1, 0⟩
⟨1, 1, 0, 0⟩
⟨1, 2, 2, 0⟩
⟨0, 0, 0, 0⟩
⟨0, 1, 2, 0⟩
⟨0, 2, 1, 0⟩
+⟨1, 0, 1, 0⟩


(0, 2)
(1, 2)
(2, 2)
(0, 1)
(1, 1)
(2, 1)
(0, 0)
(1, 0)
(2, 0)



$$\langle \mathbf{1}, \mathbf{0} \rangle\, \mathbf{x} + \langle \mathbf{0}, \mathbf{0} \rangle$$


(0, 2)
(1, 2)
(2, 2)
(0, 1)
(1, 1)
(2, 1)
$+\langle 0, 1\rangle$
(0, 0)
(1, 0)
(2, 0)
$\langle \mathbf{1}, \mathbf{0}\rangle\,\mathbf{x} + \langle \mathbf{0}, \mathbf{1}\rangle$



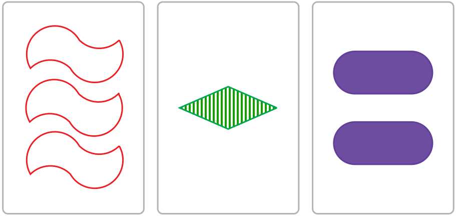
$$\langle \mathbf{1}, \mathbf{0} \rangle\, \mathbf{x} + \langle \mathbf{0}, \mathbf{2} \rangle$$

We can build $AG(n, 3)$ for any dimension n:

Points: $\langle p_1, p_2, \ldots, p_n \rangle$

Lines: $\left\{ \vec{m}x + \vec{b} \right\}$ *(always sum to* $\vec{0}$ *(mod 3)).*

SET is $AG(4, 3)$:



$\langle 0, 0, 0, 1 \rangle$ $\langle 2, 1, 1, 2 \rangle$ $\langle 1, 2, 2, 0 \rangle$

$$\langle 1, 2, 2, 1 \rangle x + \langle 0, 0, 0, 1 \rangle$$

SET: Searching for lines in an affine geometry.

Anti-SET: Avoiding lines in an affine geometry *with 3 points per line*.

Theorem

Xavier can win Anti-SET played on $AG(n, 3)$, $n > 1$.

End

Cap: A set of points that contains no line.

$m(n)$: Size of a maximal cap in n-dimensional SET.

$m(2) = 4$

Theorem: Olivia can force the game to $m(n)$ moves.

Proof:

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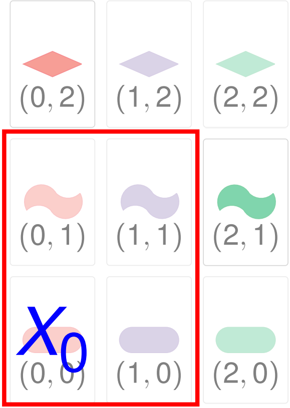
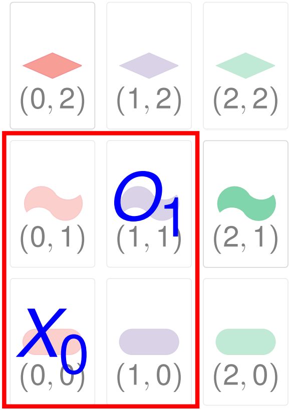
- Olivia takes every move from a maximal cap C containing X_0.



Theorem: Olivia can force the game to $m(n)$ moves.

Proof:

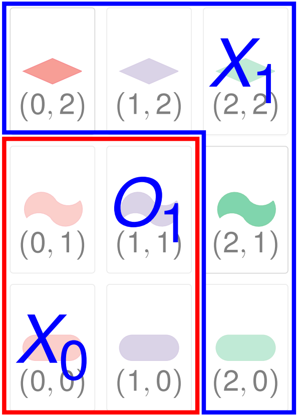
- Olivia takes every move from a maximal cap C containing X_0.
- Thus Olivia never makes a line within the cap.



Theorem: Olivia can force the game to $m(n)$ moves.

Proof:

- Olivia takes every move from a maximal cap C containing X_0.
- Thus Olivia never makes a line within the cap.
- Xavier only takes points outside C.



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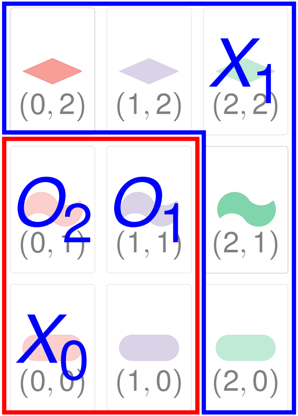

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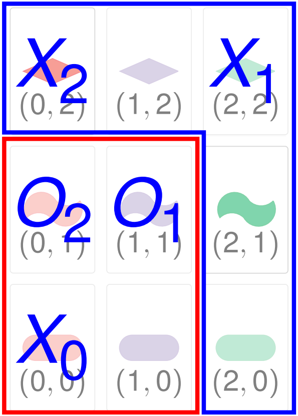

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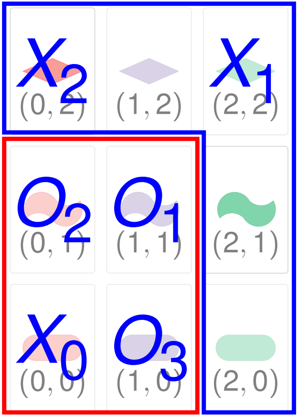

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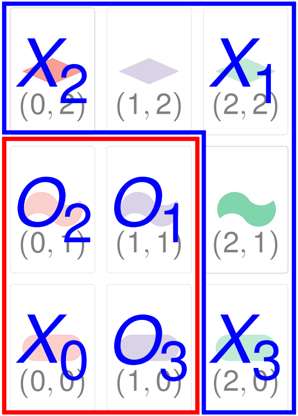

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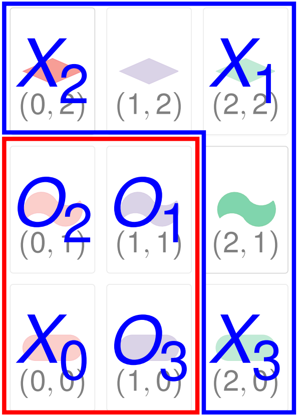
Proof:

- Olivia takes every move from a maximal cap C containing X_0.
- Thus Olivia never makes a line within the cap.
- Xavier only takes points outside C.
- Olivia can make one last move outside of C, guaranteed to lose.*

* Not obvious!



Questions?

More information:

- David Clark and George Fisk and Nurry Goren: *A variation on the game SET*.
  Involve 9 (2) (2016) 249–264.
- Benjamin Lent Davis and Diane Maclagan: *The card game SET*.
  Mathematical Intelligencer 25 (3) (2003) 33–40.
- Maureen T. Carroll and Steven T. Dougherty: *Tic-Tac-Toe on a finite plane*.
  Mathematics Magazine 77 (4) (2004) 260–274.