Nonequilibrium Kondo impurity: Perturbation about an exactly solvable point

Kingshuk Majumdar
Department of Physics and National High Magnetic Field Laboratory, University of Florida, 215 Williamson Hall,
Gainesville, Florida 32611

Avraham Schiller
Department of Physics and National High Magnetic Field Laboratory, University of Florida, 215 Williamson Hall,
Gainesville, Florida 32611
and Department of Physics, The Ohio State University, Columbus, Ohio 43210-1106

Selman Hershfield
Department of Physics and National High Magnetic Field Laboratory, University of Florida, 215 Williamson Hall,
Gainesville, Florida 32611

(Received 31 July 1997; revised manuscript received 14 October 1997)

We perturb about an exactly solvable point for the nonequilibrium Kondo problem. In each of the three
independent directions in parameter space, the differential conductance evolves smoothly as one goes away
from the solvable point, and the lowest-order correction contains the logarithm of the bandwidth, or cutoff.
Perturbing towards physically realistic exchange couplings yields differential-conductance curves which more
closely resemble experimental data than at the solvable point. The leading coefficient which describes the
low-temperature and low-voltage scaling changes as one perturbs away from the solvable point, indicating
nonuniversal behavior; however, it is restored to the solvable-point value in the limit of an infinite bandwidth.

I. INTRODUCTION

Recently, there has been a resurgence of interest in ex-
actly solvable points for many-body problems in condensed
matter physics. This is due in part to the discovery of new
physical systems and in part to the discovery of new solvable
points. For example, there is strong experimental evidence
for the realization of clean one-dimensional interacting elec-
tron systems in fractional quantum Hall systems\[12\] and quan-
tum wires,\[3,4\] as well as for tunneling through a single mag-
netic impurity.\[5,6\] Some of the models used to describe both
of these phenomena have special points in their parameter
space where a simple analytic solution can be found to the
many-body problem. Some of the new solvable points which
have been discovered recently are the Emery-Kivelson line
for the two-channel Kondo model,\[7\] the $g = \frac{1}{2}$ point for static
impurity scattering in a Luttinger liquid,\[8\] and three new Tou-
louse points for the generalized Anderson impurity model.\[9\]

Besides being exact, one of the main advantages of solv-
able points is that one can easily compute experimentally
observable quantities, e.g., the susceptibility. Historically,
calculations of observables for exact solutions and exactly
solvable points have been done in equilibrium or linear re-
sponse; however, recently there have been some solutions for
nonequilibrium problems.\[10,11\] This paper concerns one of
those solutions, namely, tunneling through a magnetic impu-
ity connected to two leads.\[12\]

The problem of tunneling through a magnetic impurity
has a long history. Zero-bias anomalies associated with tun-
neling through magnetic impurities were first discovered\[13\] in
the early 1960’s. These and later experiments\[14\] showed the
characteristic logarithmic singularities of the Kondo effect.

Shortly after the original experiments, there were perturba-
tive theories\[15\] which explained all of the qualitative features
of the experiments; however, they were not able to get to the
low-temperature, strong-coupling regime of the Kondo ef-
fect. With the present interest in quantum dots, almost all the
techniques of modern many-body physics have been applied
to this problem.\[16\] To date, the only exact result on it beyond
linear response is due to an exactly solvable point,\[12\] which
generalizes the Toulouse\[17\] and Emery-Kivelson\[7\] solutions of
the equilibrium Kondo problem.

Using this solvable point for the nonequilibrium Kondo
problem, a host of observables were computed: electrical
current, spin current, current noise, and even time-dependent
response. In the case of the differential conductance, which
is the most widely studied property, the solvable point shows
all the qualitative features of the experiments: there is a reso-
nance at zero bias—this resonance splits in an applied mag-
netic field—the temperature and voltage dependences show
 correct Fermi-liquid behavior. Furthermore, assuming uni-
versality, one can actually determine the exact scaling func-
tion for the differential conductance at low temperature and
voltage.\[12\]

Although one can obtain all this information from the
solvable point, it is still only one point in the parameter
space. There is no reason to believe that the experimental
conditions correspond to this point. Thus, a natural question
to ask is what happens as one goes away from the solvable
point? This is the question which we address in this paper. In
particular, we perturb away from the solvable point to lowest
order in all possible directions in parameter space. The ques-
tions which we ask are (i) is the perturbation away from the
solvable point smooth and nonsingular? (ii) Does the quali-
tative behavior of the differential conductance change? (iii) Does the quantitative behavior in the scaling regime change?

The organization of the rest of the paper is as follows. In the next section we describe the model. The solvable point is reviewed in Sec. III, and in Sec. IV the calculation of the perturbation away from the solvable point is presented. The results of the calculation are given in the discussion of Sec. V, and summarized in Sec. VI.

II. MODEL

The physical system we consider consists of left (L) and right (R) leads of noninteracting electrons, which interact via an exchange coupling with a spin-1/2 magnetic impurity placed in between the two leads. There are two parts to the Hamiltonian: the kinetic energy of the conduction electrons \( \mathcal{H}_0 \) and the interaction with the impurity spin \( \mathcal{H}_I \). By considering (i) a constant density of states in the wide-band limit and (ii) constant exchange couplings with the impurity spin, one can reduce the Hamiltonian to an effective one-dimensional model. The conduction electrons in the left or right leads of noninteracting electrons, which interact via applied magnetic field, which produces a Zeeman splitting of cal potentials is \( V \), and summarized in Sec. VI.

The transverse couplings \( J_z^{LL} = J_z^{RR} = J_z^{LR} = 0 \) can take arbitrary values at the solvable point. The purpose of this paper is to perturb away from the solvable point by relaxing each of the three conditions listed above.

III. OVERVIEW OF SOLVABLE POINT

We begin by briefly reviewing the solution at the solvable point, and by introducing the nonequilibrium Green functions that will later be used in the perturbation theory about this point. The approach presented here follows closely the Emery-Kivelson solution of the two-channel Kondo model.

The first step in reducing \( \mathcal{H}_I \) and \( Y_0 \) to quadratic forms is to bosonize the fermion fields \( \psi_{\alpha}(x) \) by introducing four separate boson fields \( \Phi_{\alpha}(x) \), one for each fermion field. These are used in turn to construct four new boson fields \( \Phi_{\alpha}(x) \), corresponding to collective charge (c), spin (s), flavor (f), and spin-flavor (sf) excitations. The flavor modes measure the charge-density difference between the left and right leads, while the spin-flavor modes correspond to the difference in spin densities. After a canonical transformation is performed, four new fermion fields \( \bar{\psi}(x) \) are introduced by refermionizing each of the \( \Phi_{\alpha}(x) \) fields. Finally, the impurity spin, which has been mixed by the canonical transformation with the conduction-electron spin degrees of freedom, is represented in terms of two Majorana (real) fermions \( \tilde{a} = -\sqrt{2} \tau_y \) and \( \tilde{b} = -\sqrt{2} \tau_z \). The Majorana fermions satisfy conventional anticommutation relations, except that \( \tilde{a}^2 = \tilde{b}^2 = \frac{1}{2} \). This distinguishes them from ordinary fermions. Their anticommutation with the \( \bar{\psi}(x) \) fields is guaranteed by attaching appropriate phase factors to the latter fields.

The end points of these manipulations are two quadratic operators for \( \mathcal{H} \) and \( Y_0 \):
Here $\psi_{\mu,k}$ are the Fourier transforms of $\psi_\mu(x)$; $a$ is an ultraviolet momentum cutoff, corresponding to a lattice spacing, $L$ is the size of the system, and $\epsilon_k$ is equal to $\hbar v_F k$. Relaxation of any of the three conditions listed in Eqs. (5)–(7) generates additional terms in the Hamiltonian, which are no longer quadratic in fermion operators [see Eqs. (23)–(25)]. These terms are the focus of the present paper.

Equations (8)–(9) are just a particular type of resonant-level model, in which flavor and spin-flavor conduction fermions are coupled to two Majorana fermions. They can be solved by any number of conventional techniques. We use the nonequilibrium Green function approach, whose basic ingredients are the Majorana Green functions

\[
G^>_{\alpha\beta}(t,t') = \langle \hat{a}(t) \hat{b}(t') \rangle, \tag{10}
\]

\[
G^<_{\alpha\beta}(t,t') = \langle \hat{a}(t') \hat{b}(t) \rangle, \tag{11}
\]

\[
G^{r,a}_{\alpha\beta}(t,t') = \mp i \theta(\pm t \mp t')(\{\hat{a}(t),\hat{b}(t')\}). \tag{12}
\]

Here $\alpha, \beta$ are either $a$ or $b$, while upper and lower signs in Eq. (12) correspond to retarded ($r$) and advanced ($a$) Green functions, respectively. The curly brackets in Eq. (12) denote the anticommutator. For convenience, we represent the Majorana Green functions in terms of \(2 \times 2\) matrices, with the convention that indices 1 and 2 correspond to $a$ and $b$, respectively.

Because the Hamiltonian of Eq. (8) is quadratic, one can sum the perturbation expansion to all orders, and obtain exact analytic expressions for the Majorana Green functions. These feature the energy scales

\[
\Gamma_a = \frac{1}{16\pi a^2 \hbar v_F} \left[ 4(J^{LR}_\perp)^2 + (J^{LL}_\perp - J^{RR}_\perp)^2 \right], \tag{13}
\]

\[
\Gamma_b = \frac{1}{16\pi a^2 \hbar v_F} (J^{LL}_\perp + J^{RR}_\perp)^2, \tag{14}
\]

which determine the width of the various spectral functions, and thus play the role of Kondo temperatures at the solvable point. The conventional, single-channel Kondo effect is best described by the case $\Gamma_a = \Gamma_b$, in which only a single energy scale emerges. It is also useful to introduce the combinations

\[
\Gamma_1 = \frac{1}{4\pi a^2 \hbar v_F} (J^{LR}_\perp)^2, \tag{15}
\]

\[
\Gamma_2 = \frac{1}{16\pi a^2 \hbar v_F} (J^{LL}_\perp - J^{RR}_\perp)^2, \tag{16}
\]

which enter physical quantities such as the current.

$\Gamma_a$ and $\Gamma_b$ can be interpreted as the Kondo temperatures for screening of the $\hat{a}$ and $\hat{b}$ Majorana fermions, which represent the two halves of the impurity-spin degree of freedom. The two-channel limit of Emery and Kivelson is recovered when one Kondo scale vanishes, in which case one Majorana fermion is unscreened down to zero temperature. This is reflected in the residual entropy at zero temperature and a logarithmically divergent response to a local magnetic field. Fermi-liquid characteristics are restored once both $\Gamma_a$ and $\Gamma_b$ are nonzero.

Switching to energy representations and assuming a wide-band limit, the retarded and advanced Green functions at the solvable point are given by

\[
G^{r,a}(\epsilon) = \frac{1}{(\epsilon \mp i \Gamma_a)(\epsilon \pm i \Gamma_b) - (\mu_B g_s H)^2}
\times \left[ \begin{array}{c} \epsilon \pm i \Gamma_b \\ -i \mu_B g_s H \\ i \mu_B g_s H \\ \epsilon \pm i \Gamma_a \end{array} \right]. \tag{17}
\]

The greater and lesser Green functions are determined from the matrix products

\[
G^{>,<}(\epsilon) = G'(\epsilon) \Sigma^{>,<}(\epsilon) G^a(\epsilon), \tag{18}
\]

in which

\[
\Sigma^{>,<}(\epsilon) = \left[ \begin{array}{cc} 2\Gamma_1 f_{\text{eff}}(\mp \epsilon) + 2\Gamma_2 f(\mp \epsilon) & 0 \\ 0 & 2\Gamma_1 f(\mp \epsilon) \end{array} \right]. \tag{19}
\]

are the greater (upper signs) and lesser (lower signs) self-energies. $f_{\text{eff}}(\epsilon)$ is an effective distribution function that depends explicitly on the the applied bias

\[
f_{\text{eff}}(\epsilon) = \frac{1}{2} [ f(\epsilon + eV) + f(\epsilon - eV) ] . \tag{20}
\]

In equilibrium it reduces to the ordinary Fermi-Dirac distribution function $f(\epsilon)$.

The current operator in the new representation is obtained for the current, which are not of the form of Eq. (18)

\[
I(V) = \frac{e\Gamma_1}{2\pi \hbar} \int_{-\infty}^{\infty} A_a(\epsilon) \left[ f(\epsilon - eV) - f(\epsilon + eV) \right] d\epsilon, \tag{21}
\]

\[
A_a(\epsilon) = -\text{Im} \left[ \frac{\epsilon + i \Gamma_b}{(\epsilon + i \Gamma_a)(\epsilon + i \Gamma_b) - (\mu_B g_s H)^2} \right] . \tag{22}
\]

As one perturbs away from the solvable point, $I(V)$ is modified in two distinct ways. First, the spectral function $A_a(\epsilon)$ acquires additional self-energy terms, which modify the value of the integral in Eq. (21). At the same time, a new term in the Hamiltonian gives rise to additional contributions for the current, which are not of the form of Eq. (21). Hence $I(V)$ is no longer given by Eq. (21) alone. Both effects on the current are studied in detail in the following two sections.

**IV. PERTURBATION ABOUT THE SOLVABLE POINT**

As one departs from the solvable point, there are three additional terms in the Hamiltonian of Eq. (8), associated with relaxing each of the three conditions in Eqs. (5)–(7). Each of the new terms describes an interaction between the Majorana fermions and a specific combination of conduction fermions. For example, relaxation of $J^{LL}_\perp + J^{RR}_\perp = 4\pi \hbar v_F$,
Eq. (5), couples between the $\hat{a}$ and $\hat{b}$ Majorana fermions and the spin fermions. The explicit forms of the interaction terms are

$$\mathcal{H}_1 = i \left( \frac{4 \pi \hbar v_F}{\sqrt{2}} - j^{LL} - j^{RR} \right) \sum_{k,k'} : \tilde{\psi}_j^{\dagger} \tilde{\psi}_j :,$$

$$\mathcal{H}_2 = - i \left( \frac{j^{LL} - j^{RR}}{2 \hbar} \right) \sum_{k,k'} : \tilde{\psi}_{sf,k}^{\dagger} \tilde{\psi}_{sf,k} :,$$

$$\mathcal{H}_3 = i \left( \frac{j^{LR}}{2 \hbar} \right) \sum_{k,k'} \left( \tilde{\psi}_{j,k}^{\dagger} \tilde{\psi}_{sf,k} - \tilde{\psi}_{sf,k}^{\dagger} \tilde{\psi}_{j,k} + \tilde{\psi}_{sf,k}^{\dagger} \tilde{\psi}_{sf,k} - \tilde{\psi}_{j,k}^{\dagger} \tilde{\psi}_{j,k} \right),$$

where colons indicate normal ordering with respect to the unperturbed Fermi sea, i.e., $: \tilde{\psi}_{\mu,k}^{\dagger} \tilde{\psi}_{\mu,k} : = \tilde{\psi}_{\mu,k}^{\dagger} \tilde{\psi}_{\mu,k} - \delta_{\mu,k} \theta(\tau - k).$

In this section, we derive the lowest nonvanishing order corrections to the current due to Eqs. (23)–(25). In the case of $\mathcal{H}_1$ this means second order in $(j^{LL} + j^{RR} - 4 \pi \hbar v_F)$, while $\mathcal{H}_2$ and $\mathcal{H}_3$ enter the current already at first order. Combined contributions involving more than one term are of higher order in the deviations from the solvable point, which allows us to treat each term in Eqs. (23)–(25) as a separate perturbation. As we shall see, the true perturbation parameters involve not only the deviations from the solvable point, but also the logarithm of the band width.

**A. Perturbation about $j^{LL} + j^{RR} = 4 \pi \hbar v_F$**

We begin with perturbing about $j^{LL} + j^{RR} = 4 \pi \hbar v_F$, corresponding to the Hamiltonian term $\mathcal{H}_1$. Since $\mathcal{H}_1$ is free of flavor-fermion operators, it does not modify the current operator. Hence $I(V)$ is still given by Eq. (21), only with a modified spectral function $A_s(e)$. Introducing the dimensionless parameter

$$\lambda_1 = 1 - (j^{LL} + j^{RR})/4 \pi \hbar v_F,$$

we wish to expand the Majorana-fermion self-energy to lowest nonvanishing order in $\lambda_1$. The first-order correction is zero, as the expectation value of $\mathcal{H}_1$ vanishes at the solvable point. The diagrams for the second-order contributions are depicted in Fig. 1. For a zero magnetic field, only the diagram in Fig. 1(a) survives. The diagram in Fig. 1(b) is proportional to the impurity magnetization, $(\tau \bar{\gamma}) = - i \bar{\gamma} \bar{\delta} a$, and does not contribute for a zero magnetic field.

Since the diagram in Fig. 1(a) involves convolutions in energy space, it is more convenient to evaluate it in real time $t$. This gives rise to the matrix self-energy

$$\Sigma_1^{\sigma}(t) = (\lambda_1 \hbar v_F)^2 \sigma^\dagger \left[ g_{s}^{\sigma}(t) g_{s}^{\sigma}(t) G_{s}^{\sigma}(t) \right] - g_{s}^{\sigma}(t) g_{s}^{\sigma}(t) G_{s}^{\sigma}(t) - g_{s}^{\sigma}(t) g_{s}^{\sigma}(t) G_{s}^{\sigma}(t) \sigma^\dagger,$$

in which the two $\sigma^\dagger$ Pauli matrices account for the “off-diagonal” coupling between $\hat{a}$ and $\hat{b}$ within $\mathcal{H}_1$. The scalar functions $g_{s}^{\sigma}(t)$ and $g_{s}^{\sigma}(t)$ are the unperturbed Green functions for the local spin-fermion degree of freedom, which have simple zero-temperature forms

$$g_{s}^{\sigma}(t) = \frac{i}{2 \pi v_F (t + \imath \eta)},$$

$$g_{s}^{\sigma}(t) = \frac{i \eta}{\pi v_F (t^2 + \eta^2)} \theta(\pm t).$$

At finite temperature $T$ Eq. (28) is replaced with a more complicated expression, but Eq. (29) remains unchanged.

Two points are noteworthy with regard to Eqs. (27)–(29). First, contrary to the solvable point, one must maintain a finite band width $D = \hbar v_F/a$ when perturbing away from the solvable point. This introduces a short-time cutoff $\eta = \hbar D$, which enters the spin-fermion Green functions of Eqs. (28)–(29). For consistency with the cut-off scheme used in bosonization, we take an exponential cutoff for the conduction-electron degrees of freedom, corresponding to a density of states per unit length of the form

$$\rho(e) = \frac{1}{2 \pi \hbar v_F} e^{-|e|/D}.$$

The second point to notice is that, at zero temperature, $G_{s}^{\sigma}(t)$ have closed-form analytical expressions in terms of the exponential integral function $\text{Ei}(z)$. Hence $\Sigma_1^{\sigma}(t)$ can be evaluated at $T=0$ without resorting to any numerical integration. The Fourier transforms $\Sigma_1^{\sigma}(\epsilon)$, on the other hand, require a single integration at $T=0$, as does the evaluation of $\Sigma_1^{\sigma}(t)$ for $T>0$.

For a nonzero magnetic field, one must also consider the diagram in Fig. 1(b), which is readily evaluated at zero temperature to be

$$\Sigma_2^{\sigma}(t) = 2 \delta(t) \left( \lambda_1 \hbar \right)^2 / \eta M(H) \sigma^\dagger.$$

Here $M(H) = - i G_{0b}(t=0)$ is the impurity magnetization at the solvable point, while the $\delta$ function accounts for the instantaneous nature of $\Sigma_2$. To obtain the correction to the tunneling current, one needs to substitute $\Sigma_1 + \Sigma_2$ into the retarded Green function, extract the leading-order correction to the spectral function of the $\hat{a}$ Majorana fermion, and insert the latter into Eq. (21). Although Eq. (21) is written in energy space, it is advantageous to implement this procedure in real time $t$. To see this
we note that the leading-order-in-$\lambda_1$ correction to the \( a \) Majorana spectral function is given by
\[
-\text{Im}[G'(e)\Sigma'(e)G'(e)]_{aa},
\]
where \( G'(e) \) is taken from Eq. (17), and \( \Sigma'(e) \) is the Fourier transform of
\[
\Sigma'(t) = \Sigma'_R(t) + \Sigma'_L(t).
\]
Defining the auxiliary \( 2 \times 2 \) matrix function
\[
h_{a\beta}(t) = \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} e^{-i\epsilon t/\hbar} G_{a\beta}^r(\epsilon) G_{a\beta}^r(\epsilon)
\]
\[
\times [f(\epsilon - eV) - f(\epsilon + eV)],
\]
we express the change in the tunneling current as
\[
\delta I(V) = -\frac{e\Gamma_1}{\hbar^2} \text{Im} \int_{-\infty}^{\infty} \text{Tr}[\Sigma'(t)h(-t)] dt.
\]

Similar to \( \Sigma'_R(t) \), also \( h(t) \) has a closed-form expression at zero
temperature in terms of the exponential integral function \( E_1(z) \). Hence Eq. (35) is evaluated at \( T=0 \) using just a
single numerical integration, the results of which are described in the following section.

Of particular interest is the dependence of Eq. (35) on the
band width \( D \). Specifically, we are interested in the limit \( \Gamma_D^a, \Gamma_D^b \ll D \) —
corresponding to a band width much larger than the Kondo scales — for which the low-energy physics
of the Kondo Hamiltonian is expected to be universal. While \( D \) at the solvable point can be taken to be infinite, here it explicitly enters the correction to the current. Our objective is to analytically extract the leading asymptotic dependence of \( \delta I \) on \( D \). To this end, we restrict ourselves in the remainder of this subsection to zero temperature.

Equation (35) contains two contributions, one coming from the \( \Sigma'_R(t) \) component of \( \Sigma'(t) \), and the other coming from the \( \Sigma'_L(t) \) component. The latter contribution is simply equal to
\[
-2M(H) \frac{\Gamma_1 e\lambda_1^2}{\eta} \text{Im}[\text{Tr}[\sigma^3 h(0)]],
\]
which diverges linearly with the band width. The \( \Sigma'_R(t) \) contribution is less straightforward to obtain, as it requires an integral over \( t \). Recognizing that the singular dependence of the integrand on the band width comes from the short-time behaviors of \( g^{s}(\pm t) \) and \( g^{<}(t) \), Eqs. (28)–(29), we expand \( h(t), G^{>;<}(t), \) and \( G^r(t) \) about \( t=0^+ \). (Notice that the integration over \( t \) is restricted to \( t>0 \).) This allows one to extract the leading asymptotic dependence of the integral on the band width \( D \), for voltages and fields much smaller than \( D \). The dominant term in this expansion, which comes from setting \( t=0^+ \) in each of the above matrices, precisely cancels Eq. (36). The next-order term, which is logarithmic in \( D \), yields the leading asymptotic \( D \) dependence of \( \delta I(V) \), while all other terms are regular as \( D \to \infty \). For the differential conductance \( G(V) = dI/dV \) we thus obtain the following leading asymptotic correction to the solvable-point curve, valid for voltages and fields much smaller than \( D \):
\[
\delta G(V) = \lambda_1^2 \frac{2e^2\Gamma^2}{\pi\hbar} \ln(\Gamma/D) \left[ \frac{(\mu_{bg}h + eV)(2\mu_{bg}h + eV)}{[(\mu_{bg}h + eV)^2 + \Gamma^2]^2} \right. 
\]
\[
+ \frac{(\mu_{bg}h - eV)(2\mu_{bg}h - eV)}{[(\mu_{bg}h - eV)^2 + \Gamma^2]^2} \right].
\]

Here we have set for conciseness \( \Gamma_1 = \Gamma_D^a = \Gamma_D^b = \Gamma \).

To test Eq. (37), we have compared it to a numerical evaluation of \( \delta G(V) \), based on the derivative of Eq. (35)
with respect to \( V \). Good agreement was obtained for large values of the bandwidth, indicating that the true perturbation parameter in our expansion is not \( \lambda_1 \) but rather
\[
\tilde{\lambda}_1 = \lambda_1^2 \ln(D/\Gamma).
\]

This means that \( \tilde{\lambda}_1 \) must be kept fixed in order for a meaningful \( D \to \infty \) limit to exist.

B. Perturbation about \( J_{LL}^a = J_{RR}^b \)

Next we consider the case where \( J_{LL}^a \neq J_{RR}^b \), corresponding to the Hamiltonian term \( \mathcal{H}_2 \). Similar to \( \mathcal{H}_1 \), also \( \mathcal{H}_2 \) does not contain any flavor-fermion operators, hence Eq. (21) for the current remains intact. Unlike the previous case, though, \( \mathcal{H}_2 \) modifies the \( \mu_{bg}(\epsilon) \) spectral function and thus the current already at linear order in \( J_{LL}^a - J_{RR}^b \).

At linear order in \( J_{LL}^a - J_{RR}^b \), only simple Hartree
diagram bubbles contribute to the Majorana self-energy. Overall, there are twelve different
diagrams in this category, coming from the fact that the product of any two operators from
among \( \hat{b}, \hat{a}, \psi_{sf}^\dagger, \) and \( \psi_{sf} \) can have a nonzero average at the solvable point. For a zero magnetic field, the correction to the tunneling current due to \( J_{LL}^a \neq J_{RR}^b \) is found to be
\[
\delta I(V) = \frac{e\Gamma_1}{2\pi\hbar} (\Gamma_R - \Gamma_L) \int_{-\infty}^{\infty} e^{-|\epsilon|/D} f(\epsilon) \Re[G_{bb}^r(\epsilon)] d\epsilon
\]
\[
\times \int_{-\infty}^{\infty} \Re[(G_{a\alpha}^r(\epsilon))] [f(\epsilon^+ - eV)
\]
\[
-f(\epsilon^+ + eV)] d\epsilon,
\]
where \( \Gamma_{a\alpha}(\alpha = L,R) \) is equal to \( (J_{a\alpha}^\omega)^2/16\pi a h v_F \), and
\[
\lambda_2 = \frac{J_{LL}^a - J_{RR}^b}{2\pi\hbar v_F}
\]
is the dimensionless perturbation parameter. The corresponding zero-temperature, zero-field correction to the differential conductance is equal to
\[
\delta G(V) = \lambda_2 e^2 \frac{\Gamma_1}{\pi\hbar} (\Gamma_R - \Gamma_L) (eV)^2 \Gamma_0^2 \frac{1}{[(eV)^2 + \Gamma_0^2]^2}
\]
\[
\times \Re e^{-i\epsilon_B/D} E_1(-i\Gamma_B/D).
\]
and right leads. The logarithm of the bandwidth in Eq. (41) comes from the exponential integral function $E_1$ which has the asymptotic expansion $^{23}$

$$E_1(-i\Gamma_b/D) = -\gamma - \ln(-i\Gamma_b/D) + \mathcal{O}\left(\frac{\Gamma_b}{D}\right)$$

(42)  

($\gamma$ is the Euler constant). Thus, the actual perturbation parameter in this case is

$$\tilde{\lambda}_2 = \lambda_2 \ln(D/\Gamma_b),$$

(43)

which shows that $\lambda_2$ must scale as $1/\ln(D)$ in order for a meaningful $D \to \infty$ limit to exist.

C. Perturbation about $J_z^{LR} = 0$

Finally, we consider the case where $J_z^{LR} \neq 0$, corresponding to the Hamiltonian term $H_3$. Contrary to the previous two cases, $H_3$ describes tunneling of conduction electrons between the right and left leads, and therefore modifies the current operator $\hat{I}$. In general, $\hat{I}$ is given by

$$\hat{I} = \frac{e J_z^{LR}}{2\hbar \sqrt{\pi a L}} \sum_k (\psi_{f,k}^+ + \psi_{f,k}^-) \hat{a}$$

$$- \frac{e J_z^{LR}}{2\hbar L} \sum_{k,k'} (\psi_{f,k}^+ + \psi_{f,k}^-) (\psi_{f,k'}^+ + \psi_{f,k'}^-),$$

(44)

which follows from the time derivative of the flavor-fermion number operator. For $J_z^{LR} = 0$ — the case considered thus far — the second term in Eq. (44) drops, and only the first term is left. For $J_z^{LR} \neq 0$, both terms are present.

When $\langle \hat{I} \rangle$ is computed diagrammatically for $J_z^{LR} = 0$, $\psi_{f,k}^+$ and $\psi_{f,k}^-$ in the first term of Eq. (44) can only be contracted with the $\psi_{f,k}^+ - \psi_{f,k}^-$ combination in the fourth term of Eq. (8). This gives Eq. (21) for the average current. First-order perturbation theory in $J_z^{LR}$ modifies this result in three different ways. In addition to a self-energy insertion for the Majorana Green function, as in the previous two cases, $\psi_{f,k}^+ + \psi_{f,k}^-$ in the first term of Eq. (44) can now be contracted also with the $\psi_{f,k}^- - \psi_{f,k}^+$ combination that appears in $H_3$. This gives rise to a totally new contribution for the current, which is joined by the average of the second term in Eq. (44). Overall there are eighteen different diagrams to be evaluated at first order: twelve self-energy diagrams, three diagrams for the new contraction of the first term of Eq. (44), and three diagrams for the average of the second term of Eq. (44). For a zero magnetic field, these combine to give

$$\delta I(V) = \lambda_3 \frac{e J_z^{LR} (J_z^{LL} + J_z^{RR})}{8 \pi^2 \hbar^2 v_F a}$$

$$\times \int_{-\infty}^{\infty} e^{-|e| f(e)} \text{Re}[G_{b0}^+(e)] d\epsilon$$

$$\times \int_{-\infty}^{\infty} \text{Re}[\Gamma_1 (G_{aa}^+(e'))^2 + i G_{aa}^+(e')]$$

$$\times [f(e' - eV) - f(e' + eV)] d\epsilon'.$$

(45)

with

$$\lambda_3 = \frac{J_z^{LR}}{2 \pi \hbar v_F}.$$  

(46)

The corresponding zero-temperature, zero-field correction to the differential conductance reads

$$\delta G(V) = -\lambda_3 \frac{e^2 J_z^{LR} (J_z^{LL} + J_z^{RR})}{4 \pi^2 \hbar^2 v_F a}$$

$$\times \left\{ \frac{\Gamma_a}{(eV)^2 + \Gamma_a^2} + \Gamma_f \left[(eV)^2 + \Gamma_f^2\right] \right\}$$

$$\times \text{Re} \left\{ e^{-i \Gamma_b/D} e_1 (-i \Gamma_b/D) \right\},$$

(47)

which again diverges logarithmically with the bandwidth. Thus, the true perturbation parameter is

$$\tilde{\lambda}_3 = \lambda_3 \ln(D/\Gamma_b).$$

(48)

In the presence of an applied magnetic field, Eq. (45) is supplemented by three additional terms.

V. DISCUSSION

We begin our discussion with some general remarks about the range of validity of the perturbation theory. A minimal requirement of the theory is that the total zero-bias conductance does not exceed the optimal value of $2e^2/h$. While this criterion is always met by the $\lambda_1$ perturbation, for the other two perturbations it sets upper bounds on the applicability of the linear-order approximation. Specifically, at zero temperature and zero magnetic field one obtains

$$|\lambda_2| [\ln(D/\Gamma_b) + \gamma] \leq \frac{\Gamma_a}{\Gamma_1} \frac{\Gamma_2}{\Gamma_b}.$$  

(49)

$$|\lambda_3| [\ln(D/\Gamma_b) + \gamma] \leq \frac{\Gamma_a}{2 \Gamma_b} \frac{\Gamma_1}{\Gamma_2}.$$  

(50)

Assuming the energy scales $\Gamma_a$ and $\Gamma_b$ are comparable in size (see below), Eq. (50) gives an upper bound of order unity for $|\lambda_3|$. This estimate can actually be improved by considering the large-bias limit. On the other hand, Eq. (49) is proportional to $|J_z^{LL} - J_z^{RR}|$, and therefore provides a small upper bound for $|\lambda_2|$ if $|J_z^{LL} - J_z^{RR}| \ll |J_z^{LR}|$.

Evidently, there are quite a few model parameters that enter the perturbation theory about the solvable point. Next we comment on what we expect to be the most relevant choice of model parameters. The conventional, single-channel Kondo effect is best described by the case where $\Gamma_a$ is equal to $\Gamma_b$. To see this we note that not only does this case feature just a single Kondo scale, but $\Gamma_a = \Gamma_b$ a is always generated by the Schrieffer-Wolff transformation $^{24}$ when the Hamiltonian of Eq. (2) is derived from the more fundamental Anderson model. A Schrieffer-Wolff transformation also generates equal longitudinal and transverse exchange couplings, which is obviously violated at the solvable point. Hence, in addition to setting $\Gamma_a = \Gamma_b$, $J_z^{LR}$ and $J_z^{RR}$ must
be taken to have the same sign when perturbing from the solvable point towards more realistic model parameters. In practice this means that the products $\lambda_3$ ($\Gamma_L-\Gamma_R$) and $\lambda_3 J_{\lambda b}^L$ are both positive. Below we present our results for this choice of model parameters.

The main feature of the low-temperature differential conductance of the Kondo model is a zero-bias anomaly. At the solvable point, for zero temperature and zero magnetic field, the anomaly is simply a Lorentzian. As one perturbs away from the solvable point, there is still a peak near zero bias; however, its shape is changed. This is illustrated in Fig. 2, for each of the perturbations in the realistic regime discussed above. The solid lines are the differential conductance as one perturbs away from the solvable point, and the dashed lines are the differential conductance curves at the solvable point. In all three cases, the perturbed curves lie underneath the unperturbed ones near zero bias; however, there are different reasons for this behavior in each case. In Fig. 2(b), the zero-bias conductance is reduced as one perturbs away from the solvable point in this direction. In Fig. 2(c), the zero-bias conductance remains the same, but the area under the differential conductance curve is reduced. In Fig. 2(a), both the zero-bias conductance and the area under the differential conductance curve remain the same. The reduced weight near zero bias is made up at larger voltages.

Applying a magnetic field splits the zero-bias anomaly into two peaks, separated by twice the Zeeman splitting. At the solvable point, the single Lorentzian at zero magnetic field simply splits into two symmetric Lorentzians: one at $\mu_B g_i H$ and the other at $-\mu_B g_i H$. This is not what is seen experimentally.\(^6\) There is significant broadening of the split peaks as one increases the magnetic field. To see what happens as one perturbs away from the solvable point, we compare in Fig. 3 the perturbed differential conductance curves in a finite magnetic field (solid lines) to the sum of the perturbed zero-field curves, split and shifted by $\pm \mu_B g_i H$ (dots). In Figs. 3(b) and (c), the $\lambda_2$ and $\lambda_3$ perturbations, the perturbed differential conductance is very close to the sum of the shifted zero-field curves. Indeed, in the large-band limit this relation becomes exact. On the other hand, in Fig. 3(a) the finite-field curve is different from the shifted zero-field curves. In particular, area is shifted away from the zero-bias region to higher bias. This is consistent with what is seen experimentally.\(^6\) The origin of the difference between Fig. 3(a), the $\lambda_1$ perturbation, and the other two perturbations is that only in the former case is there true inelastic scattering. The Majorana self-energies in Figs. 3(b) and 3(c) are constants, independent of energy, whereas the self-energies in Fig. 3(a) are energy dependent. Thus, in order to obtain realistic curves one needs to include inelastic scattering processes in the Majorana self-energy.

All of the above discussion concerns the qualitative features of the differential conductance curves. If there is universal behavior as $T, V \to 0$, then all points in the parameter space should probe this behavior, including the solvable point and its vicinity. In the context of the two-channel Kondo model it has been proposed that the differential conductance $G(V,T)$ at low temperature and low voltage obeys the scaling relation\(^{25}\)

$$\frac{G(0,T)-G(V,T)}{B T^p} = F(e V / k_B T).$$  

(51)

Here $B$ is a model-dependent coefficient, defined from the low-temperature expansion $G(0,T) \approx G(0,0) - B T^p$, and $F(e V / k_B T)$ is a model-independent scaling function. The exponent $p$ is equal to one half for the two-channel Kondo model,\(^{25}\) and is equal to two for the ordinary one-channel model considered in this paper.
At the solvable point, the conductance at low temperature and voltage has the form\textsuperscript{12}

\[
G(0,T) - G(V,T) = \alpha \left( \frac{eV}{k_B T} \right)^2 + \gamma \left( \frac{eV}{\Gamma_a} \right)^2 + \cdots ,
\]

(52)

where $\alpha = 3/\pi^2$ and $\gamma = -6$. The $\gamma$ term in Eq. (52) is a model-dependent correction to scaling, but the $\alpha$ term, which governs at low temperatures, is model independent. A natural question to ask is whether the coefficient $\alpha$ in Eq. (52) is indeed universal, as suggested by the solvable-point result. To study this, we perturb away from the solvable point and test whether this term changes. For the $\lambda_1$, $\lambda_2$, and $\lambda_3$ corrections described above, the tunneling current can be written in the generic form

\[
I(V,T) = \frac{e}{2\pi \hbar} \int_{-\infty}^{\infty} A(\epsilon, T) [f(\epsilon - eV) - f(\epsilon + eV)] d\epsilon,
\]

(53)

where $A(\epsilon, T)$ plays the role of a generalized spectral function [for the $\lambda_1$ and $\lambda_2$ perturbations, $A(\epsilon, T)$ is equal to $-\Gamma_1 A_{\lambda_2}(\epsilon)$]. Note that, for a zero magnetic field, there is no explicit voltage dependence to the generalized spectral function for the low-order corrections discussed above. In order to obtain the coefficient $\alpha$ in Eq. (52), the spectral function is expanded in powers of $k_B T$: $A(\epsilon, T) = a(\epsilon) + b(\epsilon)(k_B T)^2 + \cdots$. The functions $a$ and $b$ are well behaved, and their derivatives go to zero as $e \to \pm \infty$. Performing a Sommerfeld expansion of Eq. (53), the $\alpha$ coefficient is found to be

\[
\alpha = \frac{3}{\pi^2} \frac{1}{1 + 6b(0)/\pi^2a''(0)},
\]

(54)

where $a''(0)$ is the second derivative of $a(\epsilon)$. Since $b(0)$ is nonzero for all three perturbations, $\alpha$ is not universal; however, in the large-band limit it is restored to the solvable-point value of $3/\pi^2$. As noted earlier, the true perturbation parameters are the $\tilde{\lambda}_i$, which contain the logarithm of the bandwidth. This logarithmic dependence on $D$ persists in the $\lambda_i$ contributions to $a''(0)$, but is absent in $b(0)$. Consequently, if one increases the bandwidth but keeps the $\tilde{\lambda}_i$’s fixed, the ratio $b(0)/a''(0)$ vanishes logarithmically, and the solvable-point value for $\alpha$ is restored.

VI. CONCLUSION

In this paper, we have perturbed away from a solvable point of the nonequilibrium Kondo model. The differential conductance evolves smoothly as one goes away from the solvable point, in each of the three independent directions in parameter space. In all three cases, the lowest nonvanishing order change in the conductance is proportional to the logarithm of the bandwidth, or cutoff. If one perturbs towards experimentally realistic values of the exchange couplings, then the differential conductance curves more closely resemble those seen in experiments. In particular, for a finite magnetic field in the case where the self-energy contains inelastic scattering processes, there is an additional reduction in the differential conductance near zero bias, with the extra weight being shifted towards higher bias. Finally, we have studied the low-temperature and low-voltage scaling as one perturbs away from the solvable point. The leading coefficient describing the low-temperature and low-voltage scaling does change as one goes away from the solvable point, showing nonuniversal behavior; however, taking the bandwidth to infinity returns it to the value at the solvable point.

ACKNOWLEDGMENTS

This work was supported by NSF Grant No. DMR9357474, the NHMFL, and the Research Corporation. A.S. was supported in part by OSU, and by a grant from the U.S. Department of Energy, Office of Basic Energy Sciences, Division of Materials Research.


\textsuperscript{19}V. J. Emery, in Highly Conducting One-Dimensional Solids, ed-


See, e.g., Handbook of Mathematical Functions, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972), Chap. 6.
