On Two-Path Convexity in Multipartite Tournaments

Darren B. Parker* Randy F. Westhoff† Marty J. Wolf‡

MR Subject Classifications: 05C20, 52A37

Keywords: Convex sets, rank, Caratheodory number, Helly number, Radon number, hull number, multipartite tournaments

Abstract

In the context of two-path convexity, we study the *rank, Helly number, Radon number, Caratheodory number*, and *hull number* for multipartite tournaments. We show the maximum Caratheodory number of a multipartite tournament is 3. We then derive tight upper bounds for rank in both general multipartite tournaments and clone-free multipartite tournaments. We show that these same tight upper bounds hold for the Helly number, Radon number, and hull number. We classify all clone-free multipartite tournaments of maximum Helly number, Radon number, hull number, and rank. Finally we determine all convexly independent sets of clone-free multipartite tournaments of maximum rank.

1 Introduction

Convexity has been studied in many contexts. These contexts have been generalized to the concept of a *convexity space*, which is a pair $\mathcal{C} = (V, C)$, where V is a set and C is a collection of subsets of V such that $\emptyset, V \in C$ and such that C is closed under arbitrary intersections and nested unions. The set C is called the set of *convex subsets of* $\mathcal{C}$. Given a subset $S \subseteq V$, the *convex hull of* S, denoted $C(S)$, is defined to be the smallest convex subset containing S.

In the case of graphs and digraphs, V is usually taken to be the vertex set and C to be a collection of vertex subsets that are determined by paths within the graph. For a (directed) graph $T = (V, E)$ and a set $\mathcal{P}$ of (directed) paths in T, a subset $A \subseteq V$ is called $\mathcal{P}$*-convex* if, whenever $v, w \in A$, any (directed) path in $\mathcal{P}$ that originates at v and ends

*Department of Mathematics, University of Dayton, Dayton, Ohio 45469, dbparker@udayton.edu, http://academic.udayton.edu/darrenparker

†Department of Mathematics & Computer Science, Bemidji State University, Bemidji, MN 56601, rwesthoff@bemidjistate.edu

‡Department of Mathematics & Computer Science, Bemidji State University, Bemidji, MN 56601, USA, mjwolf@bemidjistate.edu, http://cs.bemidjistate.edu/mjwolf

at w can involve only vertices in A. We denote the collection of convex subsets of T by $\mathcal{C}(T)$.

In the case $\mathcal{P}$ is the set of geodesics in T, we get *geodesic convexity*, which was introduced in undirected graphs by F. Harary and J. Nieminen in [HN81]. Geodesic convexity was also studied in [CFZ02] and [CCZ01]. When $\mathcal{P}$ is the set of all chordless paths, we get *induced path convexity* (see [Duc88]). Other types of convexity include *path convexity* (see [Pfa71] and [Nie81]), *two-path convexity* (see [Var76], [EFHM72], [EHM72], and [Moo72]) and *triangle path convexity* (see [CM99]).

The most studied convexity numbers are the Helly, Radon, and Caratheodory numbers (see [JN84], [Pol95], and [CM99]). These are based on notions of independence (see [vdV93, Chap. 3]). Let $\mathcal{C} = (V, C)$ be a convexity space, and let $F \subseteq V$. Then F is *H-independent* if $\bigcap_{p \in F} C(F - \{p\}) = \emptyset$. The *Helly number* $h(\mathcal{C})$ is the size of a largest H-independent set. Equivalently, it is the smallest number h such that every finite family of convex subsets has a nonempty intersection whenever every subfamily of size h has a nonempty intersection.

The set F is *C-independent* if $C(F) \nsubseteq \bigcup_{a \in F} C(F - \{a\})$. The *Caratheodory number* $c(\mathcal{C})$ is the size of a largest C-independent set. Equivalently, it is the smallest number c such that for every $S \subseteq V$ and $p \in C(S)$, there exists $F \subseteq S$ with $|F| \leq c$ such that $p \in C(F)$.

F is *R-independent* if F does not have a *Radon partition*. That is, there is no partition $F = A \cup B$ with $C(A) \cap C(B) \neq \emptyset$. The Radon number $r(\mathcal{C})$ is the size of a largest R-independent set. This definition is not universally accepted. Often it is defined as the smallest number r in which every set of size r is R-dependent. This is one larger than in our definition. The Levi inequality (see, e.g. [vdV93, p. 169]) states that $h(\mathcal{C}) \leq r(\mathcal{C})$.

F is *convexly independent* if, for each $p \in F$, we have $p \notin C(F - \{p\})$. The rank $d(\mathcal{C})$ is the size of a largest convexly independent set. Rank is a measure of how computationally difficult it is to construct the convex subsets of a given multipartite tournament. It is an upper bound on the maximum number of vertices required to generate all convex subsets using convex hulls. In [HW96], D. Haglin and M. Wolf used the fact that the collection of two-path convex subsets in a tournament has rank 2 to construct an algorithm for computing the convex subsets of a given tournament. The algorithm runs in $O(n^4)$ serial time. They later improved this to $O(n^3)$ in [HW99].

Finally, a hull set is a set $S \subseteq V$ such that $C(S) = V$. The *hull number* $hul(\mathcal{C})$ is the size of a smallest hull set (see [ES85]).

Note that since any set that is H-, C-, or R-independent must also be convexly independent, rank is an upper bound for the Helly, Caratheodory, and Radon numbers. It is also clearly an upper bound for the hull number.

All work in tournaments has been in *two-path convexity*, where $\mathcal{P}$ is the set of all 2-paths. This is natural, as J. Varlet noted in [Var76], since if all directed paths are allowed, then the only convex subsets of strong tournaments are V and $\emptyset$. Indeed, this is true even when all paths of length three or less are allowed.

Our results extend the study of two-path convexity to multipartite tournaments. In particular, we determine maximum values of convexity invariants relative to the number of vertices and classify, when possible, all multipartite tournaments that achieve this maximum. We begin with the Caratheodory number in Section 2. In Section 3, we determine the maximum rank of general multipartite tournaments and classify all such multipartite tournaments. We then turn our attention to classifying clone-free tournaments of maximum rank, Helly number, and Radon number in Sections 4 and 5. We determine the maximum convexly independent sets of clone-free multipartite tournaments of maximum rank in Section 6.

Let $T = (V, E)$ be a digraph with vertex set V and arc set E. We denote an arc $(v, w) \in E$ by $v \to w$ and say that v dominates w. If $U, W \subseteq V$, then we write $U \to W$ to indicate that every vertex in U dominates every vertex in W. We denote by T^* the digraph with the same vertex set as T, and where (v, w) is an arc of T^* if and only if (w, v) is an arc of T. Recall that, for $p \geq 2$, T is a p-partite tournament if one can partition V into p partite sets such that every two vertices in different partite sets have precisely one arc between them and no arcs exist between vertices in the same partite set. Two vertices are clones if they have identical insets and outsets, and T is *clone-free* if it has no clones. If $u, v, w \in V$ with $u \to v \to w$, we say that v *distinguishes* the vertices u and w. Note that in a clone-free multipartite tournament, for every pair of vertices u, w in the same partite set there is at least one vertex (not in that partite set) that distinguishes u and w. If $A, B \in \mathcal{C}(T)$, we denote the convex hull of $A \cup B$ by $A \vee B$. If $v, w \in V$, we drop the set notation and write $\{v\} \vee \{w\}$ as $v \vee w$.

One can construct the convex hull of a set $U \subseteq V$ in the following way. Define $C_k(U)$ inductively by

$$C_0(U) = U, \quad C_k(U) = C_{k-1}(U) \cup \{w \in V : x \to w \to y \text{ for some } x, y \in C_{k-1}(U)\}, k \geq 1$$

Thus, $C_\infty(U) = C(U)$

To facilitate our study of bipartite tournaments, it will be helpful to consider their adjacency matrices. In the case of a bipartite tournament, however, the adjacency matrix is cumbersome. Let $P_1 = \{x_1, \cdots, x_k\}$ and let $P_2 = \{y_1, \cdots, y_\ell\}$ be the partite sets of T, a bipartite tournament. For each i and j with $1 \leq i \leq k$ and $1 \leq j \leq \ell$, let $m_{i,j} = 1$ if $x_i \to y_j$ and let $m_{i,j} = 0$ otherwise. We will call $M = (m_{i,j})$ the matrix of T. Notice that x_i distinguishes y_j and y_k if and only if $m_{i,j} \neq m_{i,k}$ and y_i distinguishes x_j and x_k if and only if $m_{j,i} \neq m_{k,i}$. In addition, identical rows or columns of the matrix of T correspond to clones.

2 Inequalities Involving the Caratheodory Number

In this section, we explore Caratheodory numbers of multipartite tournaments. The following two results show that the Caratheodory number of any multipartite tournament is at most three.

Lemma 2.1. Let T be a multipartite tournament. Suppose $U \subseteq V$ and $p \in C(U)$.

1. There is an $F \subseteq U$ with $\mid F \mid \leq 3$ such that $p \in C(F)$.

2. If U lies in a single partite set of T then there is an $F \subseteq U$ with $\mid F \mid \leq 2$ such that $p \in C(F)$.

Proof. If $|U| \leq 2$ or if $p \in U$, the result is trivial, so assume $\mid U \mid \geq 3$ and $p \notin U$. Since $p \in C(U)$ and $p \notin U$ then there is a smallest positive integer k such that $p \in C_k(U)$.

We consider two cases. First assume that U does not lie in a single partite set of T. Then there are $u, v \in U$ such that u and v lie in different partite sets of T. Since k is the smallest positive integer such that $p \in C_k(U)$ then there are $x_1, y_1 \in C_{k-1}(U)$ such that $x_1 \to p \to y_1$. Since at least one of u or v is not in the same partite set as p, then $u \to p$, $v \to p$, $p \to u$ or $p \to v$. In any case, $p \in u \vee v \vee x_1$ or $p \in u \vee v \vee y_1$ so $p \in u \vee v \vee z_1$ for some $z_1 \in C_{k-1}(U)$. Since k was chosen to be minimal, $z_1 \notin C_{k-2}(U)$ so there are $x_2, y_2 \in C_{k-2}(U)$ such that $x_2 \to z_1 \to y_2$. Since at least one of u or v is not in the same partite set as z_1, then $u \to z_1$, $v \to z_1$, $z_1 \to u$ or $z_1 \to v$. Thus $z_1 \in u \vee v \vee x_2$ or $z_1 \in u \vee v \vee y_2$, so $z_1 \in u \vee v \vee z_2$ for some $z_2 \in C_{k-2}(U)$. Since $p \in u \vee v \vee z_1$ then $p \in u \vee v \vee z_2$. Continuing in this way we can generate a sequence of vertices, $z_1, z_2, \ldots, z_k$ such that $p \in u \vee v \vee z_i$ and $z_i \in C_{k-i}(U)$ for each i. In particular, $z_k \in C_0(U) = U$ and $p \in u \vee v \vee z_k$.

Now suppose U lies in a single partite set of T. Since $C(U) \neq U$, there exist $u_1, u_2 \in U$ and $v \in V$ such that $u_1 \to v \to u_2$. Repeat the above argument with u_1 and v to create a sequence $z_1, z_2, \ldots, z_k$ such that $z_i \in u_1 \vee v \vee z_{i+1}$ for $1 \leq i \leq k-1$, $p \in u_1 \vee v \vee z_i$ and $z_i \in C_{k-i}(U)$ for each i. Let $u_3 = z_k \in U$. Then $p \in C(\{u_1, v, u_3\}) \subseteq C(\{u_1, u_2, u_3\})$. By construction, either $u_1 \to z_{k-1} \to u_3$, $u_3 \to z_{k-1} \to u_1$, $v \to z_{k-1} \to u_3$ or $u_3 \to z_{k-1} \to v$.

First assume that $u_1 \to z_{k-1} \to u_3$. If $v \to u_3$ then $v \in u_1 \vee u_3$ and $p \in u_1 \vee u_3$ so assume $u_3 \to v$. Similarly, if $z_{k-1} \to u_2$ then $z_{k-1} \in u_1 \vee u_2$ and $p \in u_1 \vee u_2$ so assume $u_2 \to z_{k-1}$. Then $u_3 \to v \to u_2$ and $u_2 \to z_{k-1} \to u_3$ imply $v, z_{k-1} \in u_2 \vee u_3$. We next show that $z_{k-2} \in u_2 \vee u_3$. If z_{k-2} is in the same partite set as U then, by construction, either $v \to z_{k-2} \to z_{k-1}$ or $z_{k-1} \to z_{k-2} \to v$. On the other hand, if z_{k-2} is not in the same partite set as U then z_{k-2} is comparable to u_1 and u_3. If $u_1 \to z_{k-2} \to u_3$ or $u_3 \to z_{k-2} \to u_1$ then $p \in C_{k-2}(U)$ which is impossible. Thus either $u_1, u_3 \to z_{k-2}$ or $z_{k-2} \to u_1, u_3$. By construction, either $z_{k-1} \to z_{k-2} \to u_1$, $u_1 \to z_{k-2} \to z_{k-1}$, $z_{k-1} \to z_{k-2} \to v$ or $v \to z_{k-2} \to z_{k-1}$. In any case we obtain $z_{k-2} \in u_2 \vee u_3$. Continuing in this way, we obtain $p \in u_2 \vee u_3$ proving (ii). The case when $u_3 \to z_{k-1} \to u_1$ is similar.

If $v \to z_{k-1} \to u_3$ then by the above argument we may assume $z_{k-1} \to u_1$. Since $v \in C(\{u_1, u_2\})$ then z_{k-1} and hence p are in $C(\{u_1, u_2\})$. The case $u_3 \to z_{k-1} \to v$ is similar. $\square$

This gives us the following.

Theorem 2.2. Let T be a multipartite tournament. Then $c(T) \leq 3$.

Since singleton subsets are convex, the Radon number of a multipartite tournament with $|V| \geq 2$ must be at least 2. If $r(T) = 2$, then every triple $\{u, v, w\} \subseteq V$ has a Radon partition, which is, without loss of generality, $\{u, v\} \cup \{w\}$. Then $w \in u \vee v$, and so $\{u, v, w\}$ is convexly dependent. Thus, $c(T) \leq d(T) = 2 = r(T)$, giving us the following.

Corollary 2.3. Let T be a multipartite tournament. Then $c(T) \leq r(T)$.

We also get an inequality between $h(T)$ and $c(T)$. We begin with the following lemma.

Lemma 2.4. Let T be a multipartite tournament. Then $h(T) = 2$ implies $c(T) = 2$.

Proof. If $h(T) = 2$, we clearly cannot have $c(T) = 1$. Let $U \subseteq V$, and let $p \in C(U)$. If U lies in a single partite set of T, then $p \in x \vee y$ for some $x, y \in U$ by Lemma 2.1(2). If U does not lie in a single partite set, then we need only show that there is $F \subset U$ with $|F| = 2$ such that $U \subseteq C(F)$. By Lemma 2.1(1), we need only consider U with $|U| = 3$. Let $U = \{x, y, z\}$. If each vertex is in a different partite set, then the graph induced by U is the transitive tournament on three vertices or a 3-cycle. In either case, there is a two-path and we let F be the set of the two endpoints of this two-path. If the vertices lie in two different partite sets, we assume without loss of generality that x and y lie in the same partite set. Thus, $x \vee z = \{x, z\}$ and $y \vee z = \{y, z\}$. Since $h(T) = 2$, $(x \vee z) \cap (y \vee z) \cap (x \vee y) \neq \emptyset$, implying that $z \in x \vee y$. This completes the proof. $\square$

This gives us the following.

Corollary 2.5. Let T be a multipartite tournament. Then $c(T) \leq h(T)$.

Proof. By Theorem 2.2 and Lemma 2.4, we need only show that if $h(T) = 1$, then $c(T) = 1$. But $h(T) = 1$ implies that any collection of nonempty convex subsets has a common vertex. Since all singleton subsets are convex, this implies $|V| = 1$, and so $c(T) = 1$. $\square$

An inequality one might expect is $c(T) \leq hul(T)$. However, as we will see in Example 5.2, the bipartite tournament B'_{2d-1} has hull number 2 and Caratheodory number 3 for $d \geq 4$, so this is not always the case.

By Theorem 2.2, the Caratheodory number of a multipartite tournament must be either 1, 2, or 3. For a multipartite tournament to have Caratheodory number 1 all subsets must be convex. This occurs precisely when T is bipartite and every vertex in one partite set dominates all the vertices in the other partite set.

Distinguishing between multipartite tournaments of Caratheodory number 2 and 3 is more difficult. The following example gives two infinite classes of bipartite tournaments of maximum Caratheodory number.

Example 2.6. For each $x \in \{0,1\}$, let $\overline{x} \in \{0,1\} - \{x\}$. For each $m \geq 1$, let $a, b_i \in \{0,1\}$ for $0 \leq i \leq 2m+1$. The matrices

$$\begin{bmatrix} a & * & \overline{a} & * & * & \cdots & * \\ b_0 & b_0 & b_1 & b_3 & b_5 & \cdots & b_{2m-1} \\ b_2 & b_2 & \overline{b}_1 & \overline{b}_2 & * & \cdots & * \\ b_4 & b_4 & * & \overline{b}_3 & \overline{b}_4 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \overline{b}_5 & \ddots & * \\ b_{2m-2} & b_{2m-2} & * & * & \ddots & \ddots & \overline{b}_{2m-2} \\ b_{2m} & \overline{b}_{2m} & * & * & \cdots & * & \overline{b}_{2m-1} \end{bmatrix}, \begin{bmatrix} a & \overline{a} & * & * & \cdots & * & * \\ b_0 & b_1 & b_3 & b_5 & \cdots & b_{2m-1} & b_{2m+1} \\ b_0 & b_1 & b_3 & b_5 & \cdots & b_{2m-1} & \overline{b}_{2m+1} \\ b_2 & \overline{b}_1 & \overline{b}_2 & * & \cdots & * & * \\ b_4 & * & \overline{b}_3 & \overline{b}_4 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & * & * \\ \vdots & \vdots & & \ddots & \ddots & \overline{b}_{2m-2} & * \\ b_{2m} & * & \cdots & \cdots & * & \overline{b}_{2m-1} & \overline{b}_{2m} \end{bmatrix}$$

represent bipartite tournaments of Caratheodory number 3. Let U consist of the vertices represented by the first two columns and the second row of the first matrix or the second and third row and first column of the second matrix. If p is the vertex represented by the first row (of either matrix), then $p \in C(U)$, but p is not in the convex hull of any proper subset of U. Thus, $c(T) \geq 3$, and so $c(T) = 3$ by Theorem 2.2.

While it may be difficult to classify the bipartite tournaments of maximum Caratheodory number, we do get the following.

Theorem 2.7. Let T be a bipartite tournament with Caratheodory number 3. Then there exist $a, \overline{a}, b_i, \overline{b_i} \in \{0,1\}$ with $a \neq \overline{a}$, $b_i \neq \overline{b_i}$ such that T has an induced bipartite subtournament with one of the following matrices.

$$\begin{bmatrix} a & a & \overline{a} & a & a & a & \cdots & a \\ b_0 & b_0 & b_1 & b_3 & b_5 & b_7 & \cdots & b_{2m-1} \\ b_2 & b_2 & \overline{b}_1 & \overline{b}_2 & b_2 & b_2 & \cdots & b_2 \\ b_4 & b_4 & b_1 & \overline{b}_3 & \overline{b}_4 & b_4 & \cdots & b_4 \\ b_6 & b_6 & b_1 & b_3 & \overline{b}_5 & \overline{b}_6 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & b_{2m-4} \\ b_{2m-2} & b_{2m-2} & b_1 & b_3 & b_5 & \ddots & \ddots & \overline{b}_{2m-2} \\ b_{2m} & \overline{b}_{2m} & b_1 & b_3 & b_5 & \cdots & b_{2m-3} & \overline{b}_{2m-1} \end{bmatrix}, \begin{bmatrix} a & \overline{a} & a & a & \cdots & a & a \\ b_0 & b_1 & b_3 & b_5 & \cdots & b_{2m-1} & b_{2m+1} \\ b_0 & b_1 & b_3 & b_5 & \cdots & b_{2m-1} & \overline{b}_{2m+1} \\ b_2 & \overline{b}_1 & \overline{b}_2 & b_2 & \cdots & b_2 & b_2 \\ b_4 & b_1 & \overline{b}_3 & \overline{b}_4 & b_4 & \cdots & b_4 \\ b_6 & b_1 & b_3 & \overline{b}_5 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \overline{b}_{2m-2} & b_{2m-2} \\ b_{2m} & b_1 & \cdots & \cdots & b_{2m-3} & \overline{b}_{2m-1} & \overline{b}_{2m} \end{bmatrix}$$

Proof. Since $c(T) = 3$, there must exist a set $U = \{u_1, u_2, u_3\}$ and $p \in C(U)$ with u_1, u_2 in the same partite set and $p \notin u_1 \vee u_2$. If $p = z_0$ is in the same partite set as u_3, then, as in the proof of Theorem 2.1, there exist vertices $z_1, \cdots, z_{2m}$ such that z_i distinguishes u_1 and z_{i+1} if i is even, z_i distinguishes u_3 and z_{i+1} if i is odd, and z_{2m} distinguishes u_1 and u_2. Also, let m be minimal with this property. We order the rows and columns of the matrix of T as follows. We let z_0 be the first row, u_3 the second row, with the remaining rows $z_2, z_4, \cdots, z_{2m}$. The first column is u_1, the second column is u_2, and the remaining columns are $z_1, z_3, \cdots, z_{2m-1}$. Denote the matrix $M = [a_{ij}]$.

Let $a = a_{11}$, $b_{2(k-2)} = a_{k1}$ for each $2 \leq k \leq m+2$, and $b_{2(\ell-3)+1} = a_{2\ell}$ for each $3 \leq \ell \leq m+2$. By the arcs already given, we have $a_{13} = \overline{a}$, $a_{ss} = \overline{b}_{2s-5}$, $a_{t(t+1)} = \overline{b}_{2t-4}$, and $a_{(2m+2)2} = \overline{b}_{2m}$, where $3 \leq s \leq m+2$ and $3 \leq t \leq m+1$. If u_1 and u_2 were to distinguish any vertex represented by a row of M besides z_{2m}, then either $p \in u_1 \vee u_2$ (if $a_{12} = \overline{a}$ or $a_{22} = \overline{b}_0$) or the minimality of m is violated. Thus, $a_{12} = a$ and $a_{r2} = b_{2(r-2)}$ for all $2 \leq r \leq m+1$. Also, if any z_i is distinguished by some u_j and z_k, where $i < k$, then the minimality of m is violated. This determines the rest of the entries of M, and thus the matrix is of the first form given in the conclusion of the theorem.

The case of z_0 in the same partite set as u_1 and u_2 is similar, which proves the theorem. $\square$

3 Convex Independence in Multipartite Tournaments

Since rank is an upper bound for the Helly, Radon, and hull numbers, it is helpful to better understand convexly independent sets.

Lemma 3.1. Let T be a multipartite tournament, and suppose A is a convexly independent set.

1. Let P_1 and P_2 be partite sets of T whose intersection with A is nonempty. Then either $(A \cap P_1) \to (A \cap P_2)$ or $(A \cap P_2) \to (A \cap P_1)$.
2. A has a nonempty intersection with at most 2 partite sets of T.

Proof. For (1), let $x \in A \cap P_1$ and $y \in A \cap P_2$. Without loss of generality, assume $x \to y$. Suppose $x' \in A \cap P_1$ and $y' \in A \cap P_2$ with $y' \to x'$. Then we have two cases. If $x \to y'$, we have $x \to y' \to x'$, which makes A convexly dependent. If $y' \to x$, then $y' \to x \to y$, again making A convexly dependent. These are both contradictions, so we must have $(A \cap P_1) \to (A \cap P_2)$.

For (2), let x, y, and z be vertices in A in three different partite sets. No matter how we orient the edges, we must have a 2-path. This makes $\{x, y, z\}$ convexly dependent, a contradiction. $\square$

We then say that A and B form a convexly independent set if $A \cup B$ is convexly independent and A and B are in distinct partite sets.

Lemma 3.1 gives us a quick proof of [Var76, Theorem 2.3]. Varlet's result refers to breadth. It turns out that breadth and rank coincide in convexity spaces [].

Corollary 3.2. Let T be a tournament, $|V| \geq 2$. Then $d(T) = 2$.

Proof. Clearly, $d(T) \geq 2$. Since each partite set of T consists of a single vertex, then Lemma 3.1(2) gives $d(T) \leq 2$ and the result follows. $\square$

A trivial upper bound for $d(T)$ is $|V|$. This bound is tight, and it is clear that $d(T) = |V|$ if and only if every subset of V is convex. Thus, we get the following.

Theorem 3.3. Let T be a multipartite tournament. Then $d(T) = |V|$ if and only if V is bipartite and every vertex in one partite set of V dominates every vertex in the other partite set.

The multipartite tournaments of maximum rank are bipartite, and tournaments have rank two, suggesting that having fewer partite sets tends to increase the rank of a multipartite tournament. This is supported by the following proposition.

Proposition 3.4. Let T be a p-partite tournament with $p \geq 3$. Then there exists a $(p-1)$-partite tournament S such that $d(T) \leq d(S)$.

Proof. Let P_1 and P_2 be partite sets of T. Define S to be the multipartite tournament with the same partite sets as T except P_1 and P_2 are put together as one partite set. The directed edges of S are the same as T except the elements of $P_1 \cup P_2$ are incomparable. For $F \subseteq V$, denote the convex hull of F in T and S by $C_T(F)$ and $C_S(F)$, respectively.

We first claim that every convex set C in T is convex in S. Suppose that $x, z \in C$, $y \in S$ with $x \to y \to z$. Then $x \to y \to z$ in T, so $y \in C$ by the convexity of C in T. Thus, C is convex in S. It follows that if $F \subseteq V$, then $C_S(F) \subseteq C_T(F)$.

Let $F \subseteq V$ be convexly independent in T, and let $x \in F$. If $x \in C_S(F - \{x\})$ then $C_S(F - \{x\}) \subseteq C_T(F - \{x\})$ implies $x \in C_T(F - \{x\})$, a contradiction. Thus, F is convexly independent in S, and so $d(T) \leq d(S)$. $\square$

In the next section, we will study the maximum rank of clone-free multipartite tournaments. It is tempting to try to use Proposition 3.4 to reduce this problem to the bipartite case. Unfortunately, it might be impossible to bring partite sets together without producing clones, as seen in the tripartite tournament in Figure 1. Merging of any two of the partite sets yields at least one pair of clones.

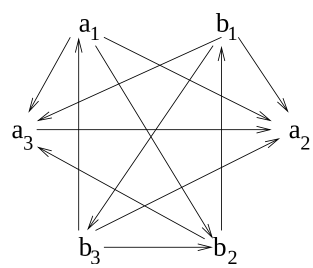

Figure 1: Merging any two partite sets yields clones

4 Maximizing Convexity Numbers in Clone-Free Multipartite Tournaments

Recall that in a clone-free multipartite tournament every pair of vertices in a given partite set is distinguished by at least one other vertex. We are particularly interested in the vertices that distinguish pairs of vertices in convexly independent sets. Given $A \subseteq V$, we define

$$D_A^{\rightarrow} = \{z \in V : z \to x \text{ for some } x \in A, y \to z \text{ for all } y \in A - \{x\}\}$$
$$D_A^{\leftarrow} = \{z \in V : z \leftarrow x \text{ for some } x \in A, z \to y \text{ for all } y \in A - \{x\}\}$$

These sets have essential properties that are used to prove our main results. The next three lemmas elucidate these properties.

Lemma 4.1. Let A and B form a convexly independent set in a multipartite tournament T, and in the case $B \neq \emptyset$ suppose $A \to B$.

1. If $|A| \geq 3$, then $D_A^{\rightarrow}$ intersects at most one partite set nontrivially. Similarly, $D_A^{\leftarrow}$ intersects at most one partite set nontrivially.

2. If $|A| \geq 2$ and $B \neq \emptyset$, then $D_A^{\rightarrow}$ is a subset of the same partite set as B. If $|B| \geq 2$ and $A \neq \emptyset$, then $D_B^{\leftarrow}$ is a subset of the same partite set as A.

3. If $|A|, |B| \geq 2$, then $D_B^{\leftarrow} \rightarrow D_A^{\rightarrow}$.

Proof. For (1), we prove the result for $D_A^{\rightarrow}$. The case of $D_A^{\leftarrow}$ follows similarly. Suppose that $z_1, z_2 \in D_A^{\rightarrow}$ with $z_1 \rightarrow z_2$. Then there exist $x_1, x_2 \in A$ with $z_1 \rightarrow x_1$ and $z_2 \rightarrow x_2$. Since $|A| \geq 3$, there exists some $x_3 \in A$ distinct from x_1 and x_2. By the definition of $D_A^{\rightarrow}$, we have $x_3 \rightarrow z_2$, so $x_3 \rightarrow z_2 \rightarrow x_2$, giving us $z_2 \in x_2 \vee x_3$. Similarly, we have $x_3 \rightarrow z_1 \rightarrow z_2$, and so $z_1 \in x_2 \vee x_3$. But $z_1 \rightarrow x_1 \rightarrow z_2$, so $x_1 \in x_2 \vee x_3$. This contradicts the convex independence of A, so (1) follows.

For (2), suppose that $z \in D_A^{\rightarrow}$ with z not in the same partite set as B. Clearly, z is also not in the same partite set as A. Since $|A| \geq 2$, there exist $x_1, x_2 \in A$ such that $x_1 \rightarrow z \rightarrow x_2$. Let $y \in B$. If $z \rightarrow y$, then $x_1 \rightarrow z \rightarrow y$ and $z \rightarrow x_2 \rightarrow y$ imply $x_2 \in x_1 \vee y$, which contradicts convex independence. If instead $y \rightarrow z$, we have $z \in x_1 \vee x_2$, and so $x_2 \rightarrow y \rightarrow z$ implies $y \in x_1 \vee x_2$, which contradicts convex independence. This implies that z and y are incomparable and are thus in the same partite set. The argument for $D_B^{\leftarrow}$ is similar.

For (3), suppose that we have $z_1 \in D_A^{\rightarrow}$, $z_2 \in D_B^{\leftarrow}$ with $z_1 \rightarrow z_2$. Since $|A|, |B| \geq 2$, then there exist $x_1, x_2 \in A$, $y_1, y_2 \in B$ such that $x_1 \rightarrow z_1 \rightarrow x_2$ and $y_1 \rightarrow z_2 \rightarrow y_2$. It follows that $z_2 \in y_1 \vee y_2$. Then $x_1 \rightarrow z_1 \rightarrow z_2$ and $z_1 \rightarrow x_2 \rightarrow y_1$ imply $x_2 \in y_1 \vee y_2 \vee x_1$, a contradiction. This proves (3). $\square$

Thus, the elements of $D_A^{\rightarrow}$ and $D_B^{\leftarrow}$ are very well-behaved. Next we explore lower bounds on $|D_A^{\rightarrow}|$ and $|D_B^{\leftarrow}|$. In the case that $|A| \geq 2$ and $B \neq \emptyset$, they turn out to be surprisingly large. They also give us insight into the structure of T.

Theorem 4.2. Let T be a clone-free multipartite tournament, and suppose that A is a convexly independent set contained in a single partite set of T. Then either $|D_A^{\rightarrow}| \geq |A| - 1$ or $|D_A^{\leftarrow}| \geq |A| - 1$. In particular, if $A = \{x_1, \cdots, x_r\}$, one can order the elements in A such that there exist $y_2, \cdots, y_r \in D_A^{\rightarrow}$ (resp., in $D_A^{\leftarrow}$) with $y_i \rightarrow x_i$ (resp., $x_i \rightarrow y_i$).

Proof. Note that if we look at A as a set of vertices in both T and T^*, then $D_A^{\leftarrow}$ in T is the same set as $D_A^{\rightarrow}$ in T^*. Thus, we need only show that $D_A^{\rightarrow} \geq |A| - 1$ in either T or T^*.

The case $r = 1$ is trivial. If $r = 2$, let y_2 be any vertex distinguishing x_1 and x_2. By relabelling x_1 and x_2, if necessary, we have $x_1 \rightarrow y_2 \rightarrow x_2$. If $r = 3$, let y_2 distinguish x_1 and x_2. By relabelling and considering T^*, if necessary, we may assume $x_1 \rightarrow y_2 \rightarrow x_2$, and that $x_3 \rightarrow y_2$. Since T is clone-free, there is some y_3 that distinguishes x_1 and x_3. By switching x_1 and x_3 if necessary, we may assume that $x_1 \rightarrow y_3 \rightarrow x_3$. It suffices to show that $x_2 \rightarrow y_3$. If $y_3 \rightarrow x_2$, then $x_1 \rightarrow y_2 \rightarrow x_2$ and $x_1 \rightarrow y_3 \rightarrow x_2$, so $y_2, y_3 \in x_1 \vee x_2$. But then $y_3 \rightarrow x_3 \rightarrow y_2$, so $x_3 \in x_1 \vee x_2$, a contradiction. Thus, $x_2 \rightarrow y_3$.

Now assume the result for $r = m \geq 3$. For $r = m + 1$, we know there exist $y_2, \cdots, y_m$ such that $y_i \to x_i$ for all $2 \leq i \leq m$ and $x_i \to y_j$ for all $i \neq j$. It is easy to see that $x_i \vee x_j = y_i \vee y_j$ for all $2 \leq i \neq j \leq m$.

For the inductive step, we need to find $y_{m+1} \in D_A^{\to}$ with $y_{m+1} \to x_{m+1}$. To this end, we first show that $x_{m+1} \to y_i$ for all $i \leq m$. Suppose that $y_i \to x_{m+1}$ for some $i \leq m$. In this case, we find that $y_i \to x_{m+1}$ for all $i \leq m$. For if there is some j for which $x_{m+1} \to y_j$, then $x_{m+1} \in y_i \vee y_j = x_i \vee x_j$, contradicting convex independence. Since $m \geq 3$, there exist $y_i, y_j \to x_{m+1}, i \neq j$. We have $x_1 \to \{y_i, y_j\} \to x_m$, and so $x_i \vee x_j = y_i \vee y_j \subseteq x_1 \vee x_m$, a contradiction. Thus, $x_{m+1} \to y_i$ for all $i \leq m$. Now we just take y_{m+1} to be a vertex distinguishing x_1 and x_{m+1}. By switching x_1 and x_{m+1}, if necessary, we can assume that $x_1 \to y_{m+1} \to x_{m+1}$.

Finally, we have to show that $x_i \to y_{m+1}$ for all $2 \leq i \leq m$. If $y_{m+1} \to x_i$, then arguments similar to the $r = 3$ case give us $x_{m+1} \in x_1 \vee x_i$, a contradiction. The lemma is proved. □

The following lemma shows that these distinguishing sets contain all vertices that distinguish vertices in A and B.

Lemma 4.3. Suppose A and B form a convexly independent set, with $A \to B$ when $A, B \neq \emptyset$.

1. If $|A| \geq 3$, then either $D_A^{\to} = \emptyset$ or $D_A^{\leftarrow} = \emptyset$. Moreover, any vertex that distinguishes two vertices in A must be in $D_A^{\to} \cup D_A^{\leftarrow}$.

2. If $|A| \geq 2$ and $B \neq \emptyset$, then any vertex that distinguishes two vertices in A is in $D_A^{\to}$.

3. If $A \neq \emptyset$ and $|B| \geq 2$, then any vertex that distinguishes two vertices in B must be in $D_B^{\leftarrow}$.

Proof. For (1), let $u \in D_A^{\to}$, $v \in D_A^{\leftarrow}$. Let $x_1, x_2 \in A$ with $u \to x_1$ and $x_2 \to v$. Then $A - \{x_1\} \to u$ and $v \to A - \{x_2\}$. We have the cases $x_1 = x_2$ and $x_1 \neq x_2$. In the case $x_1 = x_2$, ignore the x_2 and then let $x_2, x_3 \in A - \{x_1\}$. In the case $x_1 \neq x_2$, let $x_3 \in A - \{x_1, x_2\}$. In either case, $u, v \in x_1 \vee x_2$. Then $v \to x_3 \to u$ implies $x_3 \in x_1 \vee x_2$, a contradiction.

For (2), let $x, y \in A$, $z \in V$ with $x \to z \to y$, and let $w \in B$. Then $z \in x \vee y$. If $z \notin D_A^{\to}$ then there is some $v \in A - \{y\}$ such that $z \to v$. Since $z \to v \to w$, $v \in x \vee y \vee w$, which contradicts convex independence. Thus, $z \in D_A^{\to}$. We get (3) from a similar argument. □

An immediate extension of the lemma is

Corollary 4.4. Suppose A and B form a convexly independent set, and $A \to B$.

1. If $|A| \geq 3$ and $B \neq \emptyset$ then $D_A^{\leftarrow} = \emptyset$.

2. If $|B| \geq 3$ and $A \neq \emptyset$ then $D_B^{\to} = \emptyset$.

We now derive lower bounds on $|D_A^{\to}|$ and $|D_B^{\leftarrow}|$ similar to those in Theorem 4.2.

Corollary 4.5. Suppose that $A \neq \emptyset$ and $B \neq \emptyset$ form a convexly independent set, and that $A \to B$. Then $|D_A^{\rightarrow}| \geq |A| - 1$ and $|D_B^{\leftarrow}| \geq |B| - 1$.

Proof. For $D_A^{\rightarrow}$, if $|A| = 1$, the result is obvious. If $|A| = 2$ then the result follows from T being clone-free. If $|A| \geq 3$ then Corollary 4.4 implies that $D_A^{\leftarrow} = \emptyset$, and so $|D_A^{\rightarrow}| \geq |A| - 1$ by Theorem 4.2. By a similar argument $|D_B^{\leftarrow}| \geq |B| - 1$. □

The above gives us the following.

Theorem 4.6. Let $A = \{x_1, \cdots, x_m\}$ and $B = \{y_1, \cdots y_n\}$ form a convexly independent set of a multipartite tournament T with $m \geq 2$ and $n \geq 1$. Then there exist vertex subsets $U = \{u_2, \cdots, u_n\}$ and $W = \{w_2, \cdots, w_m\}$ such that $A \cup B \cup U \cup W$ induces a bipartite tournament with partite sets $A \cup U$ and $B \cup W$. The arcs are given by $A \to B$, $U \to W$, as well as

$$\{w_i \to x_i, x_j \to w_i, y_k \to u_k, u_k \to y_\ell : i \neq j, k \neq \ell\}$$

In particular, T has at least $2(m + n) - 2$ vertices.

This leads us to the main theorem of this section.

Theorem 4.7. Let T be a clone-free multipartite tournament. Then

1. $d(T)$ is at most one greater than the order of the second largest partite set in T.

2. $d(T) \leq \lfloor \frac{|V|}{2} + 1 \rfloor$.

Proof. Let A and B form a maximum convexly independent set of T with $A \to B$ when A and B are nonempty. Also, let P_1 and P_2 be the partite sets containing A and B, respectively.

For (1), if A and B are both nonempty, then Lemma 4.1(2) gives us $D_A^{\rightarrow} \subseteq P_2$ and $D_B^{\leftarrow} \subseteq P_1$. We then have $|P_1| \geq |A| + |D_B^{\leftarrow}| \geq |A| + |B| - 1 = d(T) - 1$. Thus, $d(T) \leq |P_1| + 1$. Similarly, $d(T) \leq |P_2| + 1$. In the case $B = \emptyset$, the case of $d(T) = 1$ or 2 is clear. If $d(T) \geq 3$, then Lemma 4.1(1) gives us that $D_A^{\rightarrow}$ lies in one partite set, and so does $D_A^{\leftarrow}$. Also by Theorem 4.2 either $|D_A^{\rightarrow}| \geq |A| - 1$ or $|D_A^{\leftarrow}| \geq |A| - 1$. In either case, there is a partite set $P_2 \neq P_1$ that has at least $|A| - 1$ elements. We have $d(T) = |A| \leq |P_1|$ and $d(T) = |A| \leq |P_2| + 1$, which completes the proof of (1).

For (2), note that the second largest partite set of T has at most $\frac{|V|}{2}$ vertices so that $d(T) \leq \frac{|V|}{2} + 1$ by (1). □

Corollary 4.8. Let T be a clone-free multipartite tournament, and let A and B form a maximum convexly independent set of T. Then

1. If $d(T) = \lfloor \frac{|V|}{2} + 1 \rfloor$, and if one of A or B is empty, then $|V|$ is odd.

2. Every convex subset of T is the convex hull of at most $\lfloor \frac{|V|}{2} + 1 \rfloor$ vertices.

Proof. For (1), we have $|D_A^{\rightarrow} \cup D_A^{\leftarrow}| \geq |A| - 1$ by Theorem 4.2. We then have $|V| \geq |A| + |A| - 1 = 2d(T) - 1$. This gives us $d(T) \leq \frac{|V|+1}{2}$. But this can happen only if $|V|$ is odd. The result follows.

Part (2) is a direct result of Theorem 4.7(2) and the definition of rank. □

Since rank is an upper bound for the Helly, Radon, and Caratheodory number, we get the following.

Corollary 4.9. Let T be a clone-free multipartite tournament. Then

1. $h(T)$, $r(T)$, and $hul(T)$ are at most one larger than the second largest partite set of T.

2. $h(T), r(T), hul(T) \leq \lfloor \frac{n}{2} + 1 \rfloor$.

We then say that a clone-free multipartite tournament T has *maximum rank* (resp. *maximum Helly number, maximum Radon number, maximum hull number*) if the rank (resp. the Helly number, Radon number, hull number) is $\lfloor \frac{|V|}{2} + 1 \rfloor$.

5 Classifying Clone-Free Multipartite Tournaments with Maximum Convexity Numbers

We begin this section by classifying clone-free multipartite tournaments T of maximum rank $d(T) = \lfloor \frac{|V|}{2} + 1 \rfloor$. We then use this classification to classify clone-free multipartite tournaments of maximum Helly, Radon, and hull number. As before, let A and B form a convexly independent set of T. If $A, B \neq \emptyset$, then we assume without loss of generality, that $A \rightarrow B$. For convenience, we write $d = d(T)$, so $|V| = 2d - 1$ or $2d - 2$. If $d = 1$, we just get the trivial tournament, so we may assume that $d \geq 2$. Before we commence with the classification theorems, we first consider some examples of clone-free multipartite tournaments of maximum rank.

Example 5.1. Tournaments. If T is a tournament, $d(T) \leq 2$. All tournaments with $|V| = 2$ or 3 have maximum rank. It is clear that any tournament of order 2 or 3 must also have maximum Helly, Radon, and hull number. In particular, this applies to C_3, the cyclic tournament on three vertices.

Example 5.2. Bipartite Tournaments. Let B_{2d-1} be a bipartite tournament consisting of the partite sets $P_1 = \{x_1, \cdots, x_d\}$, $P_2 = \{y_2, \cdots, y_d\}$ with $y_i \rightarrow x_i$ for all $2 \leq i \leq b$ and $x_i \rightarrow y_j$ otherwise. Note that P_1 is H-independent, R-independent, and convexly independent, so $h(B_{2d-1}) = r(B_{2d-1}) = d(B_{2d-1}) = d$. Thus, B_{2d-1} has maximum rank, Helly number, and Radon number. Also, every hull set must include x_1 and at least one of x_i or y_i for $i = 2, \ldots, d$. Thus, B_{2d-1} has maximum hull number.

Let B'_{2d-1} be the bipartite tournament consisting of the partite sets $P_1 = \{z, x_1, \cdots, x_{d-1}\}$ and $P_2 = \{y_1, \cdots, y_{d-1}\}$. The arcs are given by $P_2 \rightarrow z$, $y_i \rightarrow x_i$ for $i \geq 2$, and $x_i \rightarrow y_j$

otherwise. Note that $\{x_1, \cdots, x_{d-1}, y_1\}$ is H-independent, and thus R-independent and convexly independent. We get $h(B'_{2d-1}) = r(B'_{2d-1}) = d(B'_{2d-1}) = d$. Since $x_1 \vee z = V$, $hul(B'_{2d-1}) = 2$.

Let B_{2d-2} be a bipartite tournament consisting of the partite sets $P_1 = \{x_1, \cdots, x_{d-1}\}$ and $P_2 = \{y_1, y_2, \cdots, y_{d-1}\}$. The arcs are given by $y_i \to x_i$ for all $i \geq 2$, and $x_i \to y_j$ otherwise. Then $\{x_1, \ldots, x_{d-1}, y_1\}$ is a maximum H-, R-, and convexly independent set, so B_{2d-2} has maximum rank, Helly number, and Radon number. As with B_{2d-1}, B_{2d-2} also has maximum hull number. Notice also that $B_{2d-2} \cong B^*_{2d-2}$. This family of bipartite tournaments was previously identified by Wolf and Haglin as having exponentially many convex subsets [HW].

Example 5.3. Tripartite Tournaments. Let $T_{2d-1} = B_{2d-2} \cup \{z\}$, where $z \to B_{2d-2}$, and let $T'_{2d-1} = B_{2d-2} \cup \{z\}$, where $P_1 \to z \to P_2$ (P_1 being the partite set containing A and P_2 the partite set containing B). The maximum convexly independent sets of B_{2d-2} are also maximum convexly independent sets of T_{2d-1} and T'_{2d-1}, so both T_{2d-1} and T'_{2d-1} are of maximum rank. In T_{2d-1}, the maximum convexly independent sets are also H- and R-independent, so T_{2d-1} has maximum Helly and Radon number. However, every convex subset of T'_{2d-1} with more than one vertex contains z. It follows that $h(T'_{2d-1}) = 2$. It is straightforward to show that $r(T'_3) = 2$ and $r(T'_{2d-1}) = 3$ for $d \geq 3$. It is also straightforward to show that T'_{2d-1} has maximum hull number. Notice that $T'_{2d-1} \cong (T'_{2d-1})^*$.

A final example is $T''_5 = B_4 \cup \{z\}$ where $z \to P_1$, $y_2 \to z$, and $z \to y_1$. The unique maximum H-, R-, and convexly independent set is $\{x_1, x_2, y_1\}$, and so T''_5 has maximum rank, Helly number, and Radon number. It also has maximum hull number.

Our classification begins with the case of $B = \emptyset$.

Theorem 5.4. Let T be a clone-free multipartite tournament of maximum rank, and let A and B form a maximum convexly independent set. If $B = \emptyset$, then either $T \cong B_{2d-1}$ or $T \cong B^*_{2d-1}$.

Proof. By Corollary 4.8(1), n must be odd. Let $A = \{x_1, \cdots, x_d\}$. Theorem 4.2 implies that, by reordering the x_i's and looking at T^* if necessary, there exists $C = \{y_2, \cdots, y_d\} \subseteq D^{\to}_A$ such that $y_i \to x_i$. Furthermore, we have that $y_2, \cdots, y_d$ are all in the same partite set (if $d = 2$, it follows trivially; the $d \geq 3$ case follows from Lemma 4.1(1)). Since $n = 2d-1$, $V = A \cup C$, and so $T \cong B_{2d-1}$ (or, if we had to take T^* to get the y_i, then $T \cong B^*_{2d-1}$). $\square$

We now pursue the case of $A, B \neq \emptyset$. The following investigates some possible convexly independent sets for B_{2d-2}.

Lemma 5.5. Suppose that A and B form a maximum convexly independent set of B_{2d-2}. Let the $x_i, y_j \in B_{2d-2}$ be as in Example 5.2.

1. For all $i \geq 2$, we cannot have both $x_i \in A$ and $y_i \in B$

2. If $A \to B$, then $x_1 \in A$ and $y_1 \in B$.

Proof. For (1), suppose that $x_i \in A$, $y_i \in B$. Since $i \geq 2$, we have $d \geq 3$. Thus, either $|A| \geq 2$ or $|B| \geq 2$. If $|A| \geq 2$, we have some $x_j \in A$, $j \neq i$. Thus, $x_j \to y_i \to x_i$, contradicting convex independence. The case $|B| \geq 2$ follows similarly.

For (2), the case of $d = 2$ is obvious. For $d \geq 3$, assume for contradiction that $x_1 \notin A$. Since each y_i dominates at most one x_j, we must have $A \subseteq P_1$ and $B \subseteq P_2$. Let $r = |A|$. This leaves $d - r - 2$ vertices among $x_2, \cdots, x_{d-1}$ that are not in A. We also have $|B| = d - r$, since A and B form a maximum convexly independent set of B_{2d-2}. One of the vertices in B can be y_1, which leaves at least $d - r - 1$ vertices to be chosen from $y_2, \cdots, y_{d-1}$. Since there are only $d - r - 2$ y_i's for which $x_i \notin A$, the pigeonhole principle implies that $y_i \in B$ and $x_i \in A$ for some $i \geq 2$. This contradicts (1). The proof for $y_1 \in B$ is similar. $\square$

We now consider the cases of $|V|$ even and $|V|$ odd separately.

Lemma 5.6. Let T be a clone-free multipartite tournament of maximum rank.

1. If $|V|$ is even, then $V = A \cup B \cup D_A^{\rightarrow} \cup D_B^{\leftarrow}$, $|D_A^{\rightarrow}| = |A| - 1$, and $|D_B^{\leftarrow}| = |B| - 1$.

2. If $|V|$ is odd and $V \neq A \cup B \cup D_A^{\rightarrow} \cup D_B^{\leftarrow}$, then there exists a unique $z \notin A \cup B \cup D_A^{\rightarrow} \cup D_B^{\leftarrow}$.

Proof. If $|V|$ is even, we have $|V| = 2d - 2$. By Corollary 4.5, $|D_A^{\rightarrow}| \geq |A| - 1$ and $|D_B^{\leftarrow}| \geq |B| - 1$. We thus have

$$\begin{aligned}|V| &\geq |A| + |B| + |D_A^{\rightarrow}| + |D_B^{\leftarrow}| \\ &\geq |A| + |B| + (|A| - 1) + (|B| - 1) = 2d - 2 = |V|\end{aligned}$$

so all inequalities must be equalities, and (1) follows.

If $|V|$ is odd, we still have $|A \cup B \cup D_A^{\rightarrow} \cup D_B^{\leftarrow}| \geq 2d - 2$. This leaves one other possible vertex z, which proves (2). $\square$

Theorem 5.7. If T is a clone-free multipartite tournament of maximum rank, and if $|V| = 2d - 2$, then $T \cong B_{2d-2}$.

Proof. The case of $|V| = 2$ is obvious. We can then assume that $|V| \geq 4$ and $d \geq 3$. Since $B_{2d-2} \cong B_{2d-2}^*$, we consider T^* if necessary. Let $A = \{x_1, x_2, \ldots, x_r\}$ and $B = \{y_1, y_2, \ldots, y_s\}$. Without loss of generality, $r \geq 2, s \geq 1$ and $r + s = d$. Assume for now that $s \geq 2$. Then by Lemma 4.1(2), Theorem 4.2, and Lemma 5.6(1), $D_A^{\rightarrow} = \{z_2, \ldots, z_r\} \subseteq P_2$ and $D_B^{\leftarrow} = \{w_2, \ldots, w_s\} \subseteq P_1$. Furthermore, $z_i \to x_i$ for $i \geq 2$ and $x_i \to z_j$ otherwise; $y_k \to w_k$ for $k \geq 2$ and $w_k \to y_l$ otherwise and $D_B^{\leftarrow} \to D_A^{\rightarrow}$ by Lemma 4.1. Thus, $P_1 = \{x_1, x_2, \ldots, x_r, w_2, \ldots, w_s\}$ and $P_2 = \{y_1, z_2, \ldots, z_r, y_2, \ldots, y_s\}$. This ordering of the vertices in P_1 and P_2 and the above arc orientations show that $T \cong B_{2b-2}$.

When $s = 1$ we have $B = \{y_1\}$ and $D_B^{\leftarrow} = \emptyset$. We similarly conclude that $T \cong B_{2d-2}$. $\square$

This brings us to the case of $|V| = 2d-1$. Recall that by Lemma 5.6(2), there is at most one vertex $z \notin A \cup B \cup D_A^{\rightarrow} \cup D_B^{\leftarrow}$. When such a z exists, consider the subtournament T' induced by $V' = V - \{z\}$. Then $|V'| = 2d-2$ and A and B form a maximum convexly independent set of T'. Thus, $d(T') = d$, so $T' \cong B_{2d-2}$ by Theorem 5.7. Thus, T has at least two partite sets P_1 and P_2 with $P_1 \supseteq \{x_1, \cdots, x_{d-1}\}$, $P_2 \supseteq \{y_1, \cdots, y_{d-1}\}$ with arcs as in Example 5.2, and z is the only other vertex in T.

Lemma 5.8. Let T be a clone-free multipartite tournament with $d(T) = d \geq 3$ and $|V| = 2d-1$. Let P_1 and P_2 be partite sets of T, and let A and B form a convexly independent set with $A \subseteq P_1$, $B \subseteq P_2$ as above. Finally, assume that $z \notin A \cup B \cup D_A^{\rightarrow} \cup D_B^{\leftarrow}$.

1. If $z \notin P_2$, then $z \to B$ or $B \to z$.
2. If $z \notin P_1$, then $z \to A$ or $A \to z$.
3. If $z \notin P_1 \cup P_2$, then we cannot have $B \to z \to A$.
4. If $z \notin P_2$, then either $z \to P_2$, $P_2 \to z$, or there exists a unique $u \in P_2$ such that $u \to z$.
5. If $z \notin P_1$, then either $z \to P_1$, $P_1 \to z$, or there exists a unique $u \in P_1$ such that $z \to u$.

Proof. For (1), note that if it were not the case that $z \to B$ or $B \to z$, then z would distinguish two vertices in B. Lemma 4.3(3) would then imply $z \in D_B^{\leftarrow}$, a contradiction. This proves (1), and (2) follows similarly.

For (3), we have $d \geq 3$, so either $|A| \geq 2$ or $|B| \geq 2$. If $u, v \in A$, $w \in B$, and if $B \to z \to A$, then $w \to z \to u$ and $z \to v \to w$, so $v \in w \vee u$, a contradiction. The case $|B| \geq 2$ follows similarly.

For (4), suppose that it is not the case that $z \to P_2$ or $P_2 \to z$. Then there exist $u, v \in P_2$ with $u \to z \to v$. For contradiction, assume that there is some $w \in P_2 - \{u\}$ with $w \to z$.

In the case $z \to B$, we have $u, w \in P_2 - B = D_A^{\rightarrow}$, and without loss of generality, $v \in B$. Then there exist $x_u, x_w \in A$ with $u \to x_u$ and $w \to x_w$. By Lemma 5.5, we have $x_1 \in A$. Thus, $x_1 \to u \to x_u$ and $u \to z \to v$, so $z \in x_1 \vee x_u \vee v$. But then $x_1 \to w \to z$ and $w \to x_w \to v$, which implies $x_w \in x_1 \vee x_u \vee v$, a contradiction.

In the case $B \to z$, we have $v \in P_2 - B = D_A^{\rightarrow}$ and without loss of generality $u \in B$. Let $x_v \in A$ with $v \to x_v$. Since $w \in P_2 - \{u\}$, either $w \in B$ or $w \notin B$. Suppose $w \in B$. Then $x_1 \to v \to x_v$ and $u \to z \to v$, so $z \in x_1 \vee x_v \vee u$. But then $x_1 \to w \to z$, so $w \in x_1 \vee x_v \vee u$, a contradiction. Next for a contradiction, suppose that $w \in P_2 - B = D_A^{\rightarrow}$. Let $x_w \in A$ with $w \to x_w$. We have $x_1 \to v \to x_v$ and $u \to z \to v$, so $z \in x_1 \vee x_v \vee u$. But then $x_1 \to w \to z$ and $w \to x_w \to u$, so $x_w \in x_1 \vee x_v \vee u$, a contradiction. Thus, $z \to P_2 - \{u\}$.

In either case, we have $u \to z$ for precisely one $u \in P_2$, and so (4) is proven. Part (5) follows similarly. □

Corollary 5.9. Let T be a clone-free bipartite tournament with $d(T) = d$, $|V| = 2d - 1$. Then T is isomorphic to either B_{2d-1}, B^*_{2d-1}, B'_{2d-1}, or $(B'_{2d-1})^*$.

Proof. Let us first attack the case of $V = A \cup B \cup D^{\rightarrow}_{A} \cup D^{\leftarrow}_{B}$. Since $|V| = 2d - 1$, then Corollary 4.5 implies that either $|D^{\rightarrow}_{A}| = |A|$ or $|D^{\leftarrow}_{B}| = |B|$. In the first case, we have $T \cong B^*_{2d-1}$, and in the second, we have $T \cong B_{2d-1}$.

In the case $V \neq A \cup B \cup D^{\rightarrow}_{A} \cup D^{\leftarrow}_{B}$, we have a unique $z \notin A \cup B \cup D^{\rightarrow}_{A} \cup D^{\leftarrow}_{B}$ by Lemma 5.6(2). If $z \in P_1$, then we cannot have $z \to P_2$ because z and x_1 would be clones. If $P_2 \to z$, then $T \cong B'_{2d-1}$. Otherwise $v \to z$ for precisely one $v \in P_2$ by Lemma 5.8(4). We cannot have $v = y_i$ for $i \geq 2$ because z would be a clone of x_i. Thus, $v = y_1$ and $z \in D^{\leftarrow}_{B}$, a contradiction. Arguments are similar if $z \in P_2$, where we get $T \cong (B'_{2d-1})^*$. □

This brings us to the main theorem.

Theorem 5.10. Let T be a clone-free multipartite tournament with $d(T) = \lfloor \frac{|V|}{2} + 1 \rfloor$. Then T is isomorphic to one of B_{2d-2}, B_{2d-1}, B^*_{2d-1}, B'_{2d-1}, $(B'_{2d-1})^*$, T_{2d-1}, T^*_{2d-1}, T'_{2d-1}, T''_5, $(T''_5)^*$, or C_3.

Proof. We have already proven the case where T is bipartite. Since A, B, $D^{\rightarrow}_{A}$, and $D^{\leftarrow}_{B}$ are all contained in two partite sets, and there is at most one other vertex, only the case of T tripartite remains. In this case, we must have $|V|$ odd, a partite set P_3 consisting of one element, z, and the bipartite tournament induced by $V - \{z\}$ is isomorphic to B_{2d-2}. Thus, we can write the other partite sets as $P_1 = \{x_1, \cdots, x_{d-1}\}$ and $P_2 = \{y_1, \cdots, y_{d-1}\}$ with $y_i \to x_i$ for $i \geq 2$ and $x_i \to y_j$ otherwise. By Lemma 5.5, $x_1 \in A$ and $y_1 \in B$.

Suppose that T is not isomorphic to any of T_{2d-1}, T^*_{2d-1}, or T'_{2d-1}. By Lemma 5.8(3), we cannot have $P_2 \to z \to P_1$ unless $d = 2$. In this case, $|V| = 3$ and so we have $T \cong C_3$. Thus, we can assume $d \geq 3$. By Lemma 5.8(4),(5), this leaves us two cases: either there exists a unique $v \in P_2$ with $v \to z$ or there exists a unique $v \in P_1$ with $z \to v$.

Suppose that there exists $v \in P_2$ with $v \to z$ and $z \to P_2 - \{v\}$. For a contradiction, suppose $v \in B$, then $B \to z$, so $B = \{v\}$. Thus, $|A| \geq 2$, and there is some $u \in D^{\rightarrow}_{A}$. Let $x_u \in A$ with $u \to x_u$. If $A \to z$, then $x_1 \to u \to x_u$ and $x_u \to z \to u$, so $z \in x_1 \vee x_u$. But then $x_1 \to v \to z$, so $v \in x_1 \vee x_u$, a contradiction. Thus, $z \to A$ by Lemma 5.8(2). But then $B \to z \to A$, contradicting Lemma 5.8(3). This leaves us with $v \in P_2 - B = D^{\rightarrow}_{A}$. Since $D^{\rightarrow}_{A} \neq \emptyset$, this implies that $|A| \geq 2$.

Let $x_v \in A$ with $v \to x_v$. As before, either $A \to z$ or $z \to A$. If $A \to z$, then since $z \to B$, $x_1 \to z \to y_1$ and $x_1 \to v \to z$, so $v \in x_1 \vee y_1$. But then $v \to x_v \to y_1$, so $x_v \in x_1 \vee y_1$, a contradiction. Thus, $z \to A$. Now, since $|A| \geq 2$ and $z \to A$, Lemma 5.8(5), implies that $z \to P_1$.

We now claim that $|A| = 2$. Suppose that $|A| \geq 3$, and let $x \in A - \{x_1, x_v\}$. Then $x_1 \to v \to x_v$, $v \to z \to x_1$, and $z \to x \to y_1$ imply $x \in x_1 \vee x_v \vee y_1$, a contradiction. Thus, $|A| = 2$. This along with Lemma 5.8(5), imply $z \to P_1$.

Suppose $|B| \geq 2$. Then there is a $y \in B - \{y_1\}$ and an $x_y \in D^{\leftarrow}_{B}$ such that $y \to x_y$. As above $z \in x_1 \vee x_v$. Then $z \to x_y \to y_1$ and $z \to y \to x_y$, so $y \in x_1 \vee x_v \vee y_1$, a contradiction. Thus $d = 3$ and $|V| = 5$, so $T \cong T''_5$.

If there exists a unique $v \in P_1$ with $z \to v$, apply the above to T^*. Then $T^* \cong T''_5$, and so $T \cong (T''_5)^*$. □

The clone-free multipartite tournaments of maximum Helly, Radon, or hull number must also have maximum rank, since rank is an upper bound for these numbers. Thus, we need only consider the multipartite tournaments in Theorem 5.10. We get the following.

Theorem 5.11. Let T be a clone-free multipartite tournament with n vertices.

1. If $h(T) = \lfloor \frac{n}{2} + 1 \rfloor$, then T or T^* is isomorphic to B_{2d-1}, B'_{2d-1}, B_{2d-2}, T_{2d-1}, T''_5 or C_3.

2. If $r(T) = \lfloor \frac{n}{2} + 1 \rfloor$, then one of T or T^* is isomorphic to B_{2d-1}, B'_{2d-1}, B_{2d-2}, T_{2d-1}, T'_5, T''_5 or C_3.

3. If $hul(T) = \lfloor \frac{n}{2} + 1 \rfloor$, then T or T^* is isomorphic to B_{2d-1}, B'_3, B_{2d-2}, T'_{2d-1} or C_3.

6 Convexly Independent Sets for Clone-Free Multipartite Tournaments of Maximum Rank

We now consider the maximum convexly independent sets of clone-free multipartite tournaments of maximum rank. The case of rank one is trivial, and in the case of rank two, we can take A to be any set of two vertices in the same partite set, or we can take A and B to be singleton sets in different partite sets. Therefore assume that $d(T) \geq 3$. Also note that, for any multipartite tournament T, convex subsets of T are identical to those of T^*.

For $w \in C(U)$ let $r_U(w)$ be the smallest nonnegative integer k such that $w \in C_k(U)$. Note that $r_U(u) = 0$ for $u \in U$. If a vertex $w \notin C(U)$, let $r_U(w) = \infty$.

Suppose that $U \subseteq V$ is not a convexly independent set. This means that there is some $x \in U$ such that $x \in C(U - \{x\})$. In particular, there is some $x \in U$ for which $k = r_{U-\{x\}}(x) \geq 1$ and $x \in C_k(U - \{x\})$ but $x \notin C_{k-1}(U - \{x\})$. This observation is useful in proving the following.

Theorem 6.1. Let T be a clone-free multipartite tournament of maximum rank $d \geq 3$. With the notation from Section 5, the sets A and B that form maximum convexly independent sets in T (with $A \to B$) are precisely the following.

1. If $T \cong B_{2d-2}$, then $A = \{x_1, x_{i_2}, \cdots, x_{i_r}\}$ and $B = \{y_1, y_{j_2}, \cdots, y_{j_s}\}$, where $r+s = d$ and $i_k \neq j_l$ for all $2 \leq k \leq r$, $2 \leq l \leq s$.

2. If $T \cong B_{2d-1}$, then $A = \{x_1, x_{i_2}, \cdots, x_{i_r}\}$ and $B = \{y_{j_1}, \cdots, y_{j_s}\}$, where $r + s = d$ and $i_k \neq j_l$ for all $2 \leq k \leq r$, $1 \leq l \leq s$.

3. If $T \cong B'_{2d-1}$, then A and B are as in (1), except when $d = 3$, we also have $A = \{y_1, y_2\}$ and $B = \{z\}$ or $A = \{y_2\}$ and $B = \{x_2, z\}$.

4. If $T \cong T_{2d-1}$ or T'_{2d-1}, then A and B are as in (1).

5. If $T \cong T''_5$, then $A = \{x_1, x_2\}$ and $B = \{y_1\}$.

Proof. For (1), we know by Lemma 5.5 that $x_1 \in A$ and $y_1 \in B$, so the elements in A are $x_i's$ and the elements in B are $y_i's$. If $i_k = j_l$ for some k, l then we have $x_1 \to y_{j_l} \to x_{i_k}$, contradicting convex independence. Thus, it suffices to show that the A and B listed above are convexly independent sets. Suppose that $y_j \in B$ makes $A \cup B$ convexly dependent. Let $U = A \cup B - \{y_j\}$, $r = r_U(y_j)$ and $r' = r_U(x_j)$. Then we must have $x_i \to y_j \to x_k$ for some $x_i, x_k \in C_{r-1}(U)$. This forces $k = j$ and $r > r'$. Similarly, since $x_j \notin A$ we must have $y_m \to x_j \to y_n$ for some $y_m, y_n \in C_{r'-1}(U)$. As before, we have $m = j$ and so $r' > r$, a contradiction. Thus, A and B form a convexly independent set. Part (2) follows similarly.

For (3), we must first consider the case of $z \in A \cup B$. Let $P_1 = \{z, x_1, \cdots, x_{d-1}\}$ and $P_2 = \{y_1, \cdots, y_{d-1}\}$ as in Example 5.2. Since P_1 is not a convexly independent set, both A and B are nonempty. Since $P_2 \to z$ and $A \to B$, we must have $z \in B$, $B \subseteq P_1$, and $A \subseteq P_2$. Clearly, $x_1 \notin B$. If $x_i \in B$ for $i > 1$, then since $x_i \to y_j$ for all $i \neq j$, we have $A \subseteq \{y_i\}$. This implies $d = 3$. Thus, $A = \{y_2\}$ and $B = \{x_2, z\}$. Otherwise, $B = \{z\}$ and $A = P_2$. In this case, it suffices to prove that $d = 3$. If $d > 3$, then $y_1, y_2, y_3 \in A$. We have $y_2 \to x_2 \to y_1$ and $x_2 \to y_3 \to z$, so $y_3 \in y_1 \vee y_2 \vee z$, a contradiction. Thus, $d = 3$, $A = \{y_1, y_2\}$, and $B = \{z\}$.

In the case $A, B \subseteq V - \{z\}$, the only possible maximum convexly independent sets of T are those given in (1). We need only show that all the sets from (1) are convexly independent sets of T. But z cannot be in the convex hull of $A \cup B$, since $P_2 \to z$, so A and B need only be a convexly independent set of the bipartite tournament induced by $V - \{z\}$, which is isomorphic to B_{2d-2}. This was shown in (1).

For (4), as in Section 5, $A \cup B \cup D_A^{\rightarrow} \cup D_B^{\leftarrow}$ is contained in two partite sets each with at least two vertices. In particular, $z \notin A \cup B$. Thus, $A, B \subseteq V - \{z\}$, which induces a bipartite tournament isomorphic to B_{2b-2}. As before, we need only prove that all the sets in (1) are convexly independent sets of T. If $T \cong T_{2d-1}$, then this follows as in (1) since $z \to V - \{z\}$. If $T \cong T'_{2d-1}$, then suppose that $A \cup B$ is convexly dependent. By (1), $A \cup B$ cannot be made convexly dependent by vertices in $P_1 \cup P_2$. Thus, there must be some x_i (resp. y_i) that was brought into the convex hull of $(A \cup B) - \{x_i\}$ (resp. $(A \cup B) - \{y_i\}$) by z that could not have been brought in without z. This would occur by $y_j \to x_i \to z$ (resp. $z \to y_i \to x_j$). But since $x_1 \in A$ and $y_1 \in B$, we could just as well have gotten x_i and y_i by $y_j \to x_i \to y_1$ and $x_1 \to y_i \to x_j$. Thus, z has no effect on whether or not x_i or y_i make $A \cup B$ convexly dependent, and the result follows from (1).

For (5), we again know $z \notin A \cup B$. By Lemma 5.5, we have $x_1 \in A$ and $y_1 \in B$. Since $y_2 \to z \to y_1$, Lemma 5.8(1) implies that $y_2 \notin B$. Thus, $A = \{x_1, x_2\}$ and $B = \{y_1\}$. □

7 Open Problems

We end with three open problems related to our results.

(1) *For which multipartite tournaments do we have $h(T) = r(T) = d(T)$?* This occurred for multipartite tournaments of maximum Helly number, but did not occur for T'_{2d-1}.

(2) *For which multipartite tournaments do we have $hul(T) = d(T)$?* This occurred for multipartite tournaments of maximum hull number, but not for B'_{2d-1}, T_{2d-1}, and T''_5.

(3) *Classify all multipartite tournaments of minimum rank.* Certainly, all nontrivial tournaments have rank 2. It is natural to try to describe other multipartite tournaments of rank 2.

References

[CCZ01] G. Chartrand, A. Chichisan, and P. Zhang, *On convexity in graphs*, Cong. Numer. **148** (2001), 33–41.

[CFZ02] G. Chartrand, J.F. Fink, and P. Zhang, *Convexity in oriented graphs*, Discrete Applied Math. **116** (2002), 115–126.

[CM99] M. Changat and J. Mathew, *On triangle path convexity in graphs*, Discrete Math. **206** (1999), 91–95.

[Duc88] P. Duchet, *Convexity in graphs II. Minimal path convexity*, J. Combin. Theory, Ser. B **44** (1988), 307–316.

[EFHM72] P. Erdös, E. Fried, A. Hajnal, and E.C. Milner, *Some remarks on simple tournaments*, Algebra Universalis **2** (1972), 238–245.

[EHM72] P. Erdös, A. Hajnal, and E.C. Milner, *Simple one-point extensions of tournaments*, Mathematika **19** (1972), 57–62.

[ES85] M.G. Everett and S.B. Seidman, *The hull number of a graph*, Discrete Math. **57** (1985), 217–223.

[HN81] F. Harary and J. Nieminen, *Convexity in graphs*, J. Differential Geometry **16** (1981), 185–190.

[HW] D.J. Haglin and M.J. Wolf, *On convex subsets in multipartite tournaments*, preprint.

[HW96] ———, *On convex subsets in tournaments*, SIAM Journal on Discrete Mathematics **9** (1996), 63–70.

[HW99] ———, *An optimal algorithm for finding all convex subsets in tournaments*, Ars Combinatoria **52** (1999), 173–179.

[JN84] R.E. Jamison and R. Nowakowski, *A Helly theorem for convexity in graphs*, Discrete Math. **51** (1984), 35–39.

[Moo72] J.W. Moon, *Embedding tournaments in simple tournaments*, Discrete Math. **2** (1972), 389–395.

[Nie81] J. Nieminen, *On path- and geodesic-convexity in digraphs*, Glasnik Matematicki **16** (1981), 193–197.

[Pfa71] J.L. Pfaltz, *Convexity in directed graphs*, J. Combinatorial Theory **10** (1971), 143–152.

[Pol95] N. Polat, *A Helly theorem for geodesic convexity in strongly dismantlable graphs*, Discrete Math. **140** (1995), 119–127.

[Var76] J.C. Varlet, *Convexity in Tournaments*, Bull. Societe Royale des Sciences de Liege **45** (1976), 570–586.

[vdV93] M.L.J van de Vel, *Theory of Convex Structures*, North Holland, Amsterdam, 1993.