Multidesigns for Graph-Triples of Order 6

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MR Subject Classifications: 05C70, 05B40
Keywords: Multidecomposition, multidesign, multipacking, multicovering, graph decomposition.

Abstract

We call $T = (G_1, G_2, G_3)$ a *graph-triple of order* t if the G_i are pairwise non-isomorphic graphs on t non-isolated vertices whose edges can be combined to form K_t. If $m \geq t$, we say T *divides* K_m if $E(K_m)$ can be partitioned into copies of the graphs in T with each G_i used at least once, and we call such a partition a T*-multidecomposition*. In this paper, we study multidecompositions of K_m for graph-triples of order 6. We focus on graph-triples in which either one graph is a perfect matching or all graphs have 5 edges each. Moreover, we determine maximum multipackings and minimum multicoverings when K_m does not admit a multidecomposition.

1 Introduction

The graph decomposition problem, in which the edges of a graph are decomposed into copies of a fixed subgraph, has been widely studied (see [BHRS80], [BS77], and [Kot65]). In [AD03], A. Abueida and M. Daven extended this notion to *graph-pairs*. Given graphs G_1 and G_2 such that $G_1 \cup G_2 = K_t$, they sought complete graphs K_m with $m \geq t$ whose edges can be partitioned into copies of G_1 and G_2 using at least one copy of each graph. They called such a partition a (G_1, G_2)*-multidecomposition*.

In the same paper, the authors studied maximum *multipackings* and minimum *multicoverings* when a multidecomposition is impossible. A maximum multipacking is a partitioning of a subset of $E(K_m)$ into copies of G_1 and G_2, using at least one copy of each G_i where the number of edges outside the partition, called the *leave*, is minimum. A minimum multicovering is a collection of copies of both G_i that use all edges of K_m at least once and where the number of edges used more than once, called the *padding*, is minimum. A *multidesign* refers to a multidecomposition, a maximum multipacking, or a minimum multicovering. The authors solved the existence problem for all optimal multidesigns of K_m into graph-pairs of order 4 and 5. In [ADR05], Abueida, Daven and K. Roblee proved similar results for multidesigns of λK_m into graph-pairs of orders 4 and 5 for any value of $\lambda \geq 1$.

In this paper we define a *graph-triple* $T = (G_1, G_2, G_3)$ of order t to be a triple of non-isomorphic graphs G_1, G_2, and G_3 without isolated vertices that that *factor* K_t (i.e. $G_1 \cup G_2 \cup G_3 = K_t$). We define T-multidecompositions, T-multipackings, T-multicoverings, T-multidesigns, and the notion of T dividing a graph analogously with the graph-pair definitions.

One can show that there are no graph-triples of order $t \leq 5$. We therefore consider graph-triples of order 6. An exhaustive search shows that there are 131 such graph-triples (see Appendix B). In Section 2, we determine the sizes of the leave and padding for all optimal multidesigns of K_m into graph-triples of

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order 6 that include a perfect matching (see Theorem 2.10). In Section 3, we prove analogous results for graph-triples whose graphs have 5 edges each (see Theorem 3.12).

We list the graphs that are part of graph-triples of order 6 in Appendix A. In memory of Frank Harary, we will denote the i^{th} graph on 6 vertices with j edges and no isolated vertices with the notation H_i^j. The graphs are obtained from [HP73], where we remove graphs that cannot be part of a graph-triple of order 6. Note that the vertices are labeled a through f. If $v_k \in V(K_m)$ for $k \in \{a, b, c, d, e, f\}$, we will denote by $[v_a, v_b, v_c, v_d, v_e, v_f]$ the subgraph of K_m isomorphic to H_i^j in which each v_k plays the role of k. This will not be ambiguous as long as we specify H_i^j.

We write $V(G)$ to denote the vertex set of G and $\deg(v)$ to denote the degree of $v \in V(G)$. Further, $\Delta(G) = \max\{\deg(v) : v \in G\}$. We write $G_1 + G_2$ to denote any graph with edge set $E(G_1) \cup E(G_2)$ and kG_1 to be a graph whose edges can be partitioned into k copies of G_1. We let $V(K_n) = \mathbb{Z}_n$, and for $r \leq n$, we consider $\mathbb{Z}_r \subseteq \mathbb{Z}_n$ in the natural way. Note that $\mathbb{Z}_r$ induces a subgraph of $\mathbb{Z}_n$ isomorphic to K_r. We define $G_{r,m} = K_m - K_r$ with $V(G_{r,m}) = \mathbb{Z}_n$, and we let the vertices from which the edges of K_r are removed be $\mathbb{Z}_r$. If $m \geq 6$, we have $K_m = K_6 \cup G_{6,m}$. We can factor K_6 into any graph-triple of order 6, and so we get the following.

Lemma 1.1. Let $m \geq 6$, and let $T = (G_1, G_2, G_3)$ be a graph-triple of order 6. Suppose $G_{6,m}$ has a T-multipacking with leave L (resp. a T-multicovering with padding P). Then K_m has a T-multipacking with leave L (resp. a T-multicovering with padding P).

For other terminology used but not defined herein, see [BM79], [LR97].

2 Multidesigns for Graph-Triples (G_1, G_2, H_1^3)

In this section, we determine multidesigns of K_m for graph-triples $T = (G_1, G_2, H_1^3)$ of order 6. The multidesigns are generated recursively. We begin with a lemma.

Lemma 2.1. H_1^3 divides $K_{3,m}$ for all $n \geq 3$.

Proof. The cases $m = 3, 4, 5$ are easy to prove. Let $m = 3k + r$ with $k \geq 0$, and $r = 3$, 4, or 5. We have $K_{3,m} = K_{3,r} + kK_{3,3}$. Then H_1^3 divides $K_{3,r}$ and $K_{3,3}$, which completes the proof. □

This gives us the following.

Lemma 2.2. Let $T = (G_1, G_2, H_1^3)$ be a graph-triple of order 6, and let $m \geq 6$, $m \neq 7$. For each T-multidesign of K_m, there is a T-multidesign of K_{m+3} with the same leave or padding.

Proof. Take $\mathbb{Z}_m \subseteq V(K_{m+3})$, whose induced subgraph is K_m, which has the given T-multidesign. Without loss of generality, $\mathbb{Z}_6$ is the vertex set of $H_1^3 \cong [0, 1, 2, 3, 4, 5]$ in the T-multidesign. If $m \neq 8$, remove the edges of H_1^3, and add in $H_1^3 \cong [0, 1, m, m+1, 2, m+2]$, $[2, 3, m+1, m+2, 4, m]$, $[4, 5, m, m+2, 0, m+1]$, $[1, m, 3, m+1, 5, m+2]$, $[0, m, 2, m+1, 4, m+2]$, $[1, m+1, 3, m+2, 5, m]$, $[0, m+2, 2, m, 4, m+1]$, $[1, m+2, 3, m, 5, m+1]$. The remaining edges between $\mathbb{Z}_m$ and $\{m, m+1, m+2\}$ form a graph isomorphic to $K_{3,m-6}$, which can be filled in with copies of H_1^3 by Lemma 2.1. The leave or padding is unchanged.

What remains is the case $m = 8$. Now $\mathbb{Z}_8$ induces a K_8 in K_{11}, which has the given T-multidesign. We remove $H_1^3 \cong [0, 1, 2, 3, 4, 5]$ and insert $H_1^3 \cong [6, 8, 7, 9, 5, 10]$, $[6, 9, 7, 10, 0, 8]$, $[6, 10, 7, 8, 1, 9]$, $[8, 9, 3, 10, 4, 5]$, $[9, 10, 4, 8, 2, 3]$, $[8, 10, 2, 9, 0, 1]$, $[0, 9, 1, 8, 2, 10]$, $[3, 8, 4, 10, 5, 9]$, $[0, 10, 2, 8, 3, 9]$, $[1, 10, 4, 9, 5, 8]$ gives us a T-multidesign with the same leave or padding as that in K_8. □

Lemma 2.2 reduces our problem to determining optimal multidesigns for each congruence class modulo 3. The case $m \equiv 0 \pmod 3$ is easily disposed of by a factorization of K_6. It is different for $m \equiv 1, 2 \pmod 3$, as in those cases not every multidesign is a multidecomposition. For $m \equiv 1 \pmod 3$, we have the following.

Theorem 2.3. Let $T = (G_1, G_2, H_1^3)$ be a graph-triple of order 6.

1. T divides K_{10}.
2. If $G_1 = H_i^8$ and $G_2 = H_j^4$, then T does not divide K_7.
3. If $G_1 = H_i^7$ and $G_2 = H_j^5$, then T divides K_7 if and only if $(i, j) \in \{(4, 2), (5, 2), (5, 3), (5, 7), (6, 2), (8, 3)\}$.
4. If $G_1 = H_i^6$ and $G_2 = H_j^6$, then T divides K_7 if and only if $(i, j) \neq (1, 8)$.

Proof. For part (1), an H_1^3-decomposition of $G_{6,10}$ is $H_1^3 \cong [0, 6, 1, 7, 8, 9]$, $[7, 8, 2, 6, 3, 9]$, $[6, 7, 4, 8, 5, 9]$, $[6, 8, 0, 7, 1, 9]$, $[7, 9, 2, 8, 3, 6]$, $[6, 9, 4, 7, 5, 8]$, $[0, 8, 1, 6, 2, 7]$, $[4, 9, 3, 7, 5, 6]$, $[0, 9, 1, 8, 4, 6]$, $[2, 9, 3, 8, 5, 7]$. By Lemma 1.1, T divides K_{10}

For (2) and (3), assume T divides K_7. Then $K_7 = H_i^8 + H_j^4 + 3H_1^3$, and so $K_7 - H_i^8 - H_j^4 \cong 3H_1^3$. Thus, any vertex in $K_7 - H_i^8 - H_j^4$ must have degree 3 or less. We assume $V(H_i^8) = \mathbb{Z}_6$ and note that the vertex 6 does not appear in H_i^8.

Now we attack (2). If $(i, j) \neq (4, 3)$, then $\Delta(H_j^4) = 2$, so in $K_7 - H_i^8 - H_j^4$ we have $\deg(6) \geq 4$. But this implies that H_1^3 does not divide $K_7 - H_i^8 - H_j^4$, a contradiction. For the remaining triple $T = (H_4^8, H_3^4, H_1^3)$, assume that $\deg(0) = 1$ in H_4^8, and observe that $\Delta(H_3^4) = 3$. In $K_7 - H_4^8 - H_3^4$ we have $\deg(0) \geq 4$ or $\deg(6) \geq 4$ (or both), and thus H_1^3 does not divide $K_7 - H_4^8 - H_3^4$. This is a contradiction, and so T does not divide K_7.

For (3), the T-decompositions of K_7 with $(i, j) \in \{(4, 2), (5, 2), (5, 3), (5, 7), (6, 2), (8, 3)\}$ are given in Appendix C. If $(i, j) = (9, 4)$, we may assume that $\deg(0) = \deg(3) = 1$ in H_9^7, and we observe that $\Delta(H_4^5) = 3$. In $K_7 - H_9^7 - H_4^5$ we have $\deg(0) \geq 4$, $\deg(3) = 6$, or $\deg(6) \geq 4$, and thus F does not divide $K_7 - H_9^7 - H_4^5$. If $(i, j) \in \{(1, 1), (2, 1), (2, 5), (3, 1), (3, 6), (10, 1)\}$, then $\Delta(H_2^5) = 2$, so in $K_7 - G_1 - G_2$ we have $\deg(6) \geq 4$. Thus, F does not divide $K_7 - G_1 - G_2$, and so T does not divide K_7.

For (4), the T-multidecompositions for $(i, j) \neq (1, 8)$ are given in Appendix C. If (H_1^6, H_8^6, H_1^3) divides K_7, we can assume $H_1^6 \cong [0, 1, 2, 3, 4, 5]$. Since $\Delta(H_1^6) = \Delta(H_8^6) = 2$, the vertex 6 has degree at least 4 in $K_7 - H_1^6 - H_8^6$. Thus, the remaining edges cannot be partitioned into copies of H_1^3, and so there must be a copy of either H_1^6 or H_8^6 remaining. This is impossible if $V(H_8^6) = \mathbb{Z}_6$. Thus, without loss of generality, $H_8^6 \cong [6, 1, 0, 2, 4, 3]$. But then there are no copies of H_8^6 and a unique copy $[1, 4, 6, 0, 3, 5]$ of H_1^6 in $K_7 - H_1^6 - H_8^6$. The edges 26 and 25, remain, which cannot be part of H_1^3. □

For the remaining multidesigns of K_7, note that a (H_i^8, H_j^4, H_1^3)- multipacking can have a leave of no fewer than two edges.

Theorem 2.4. Let T be a graph-triple of order 6.

1. If $T = (H_1^6, H_8^6, H_1^3)$, then there exist T-multidesigns of K_7 whose leave and padding are both P_4.
2. If $T = (H_i^8, H_j^4, H_1^3)$, then there exist T-multidesigns of K_7 with leave $P_2 + P_2$ and padding P_2.
3. If $T = (H_i^7, H_j^5, H_1^3)$, then there exists a T-multipacking of K_7 with leave P_2 for all $(i, j) \neq (3, 6)$. If $(i, j) = (3, 6)$, we have an optimal leave of $P_3 + P_2$.
4. If $T = (H_i^7, H_j^5, H_1^3)$, then there exists a T-multicovering of K_7 with padding P_2 for $(i, j) \neq (10, 1)$. For $(i, j) = (10, 1)$, we get a padding of P_3.

Proof. For (1), we have the T-multipacking given by $H_1^6 \cong [0, 1, 2, 3, 4, 5]$, $H_8^6 \cong [0, 2, 1, 3, 5, 6]$, and $H_1^3 \cong [0, 4, 1, 6, 2, 5]$, $[0, 3, 1, 4, 5, 6]$. The leave is $\{2, 4\}$, $\{4, 6\}$, $\{3, 6\}$, which can be part of a T-multicovering with a 3-edge padding. This is clearly optimal.

The remaining multidesigns are listed in Appendix C. Part (2) follows easily, and so it suffices to prove that for $T = (H_i^7, H_j^5, H_1^3)$, we have neither a T-multipacking with leave P_2 for $(i, j) = (3, 6)$ nor a T-multicovering with leave P_2 for $(i, j) = (10, 1)$. These can be proven using arguments similar to those in Theorem 2.3(2) and (3). □

We now consider the case $m \equiv 2 \pmod 3$. We begin with the case $T = (H_i^6, H_j^6, H_1^3)$, in which a T-multidecomposition is impossible. In Appendix C, we determine a T-multipacking with leave P_2 for each graph-triple $T = (H_i^6, H_j^6, H_1^3)$. Note that by adding in the remaining edge and two other edges disjoint to the first, we get T-multicoverings with leaves P_3 and $P_2 + P_2$. This gives us the following.

Theorem 2.5. Let $T = (H_i^6, H_j^6, H_1^3)$ and $m \equiv 2 \pmod 3$. Then K_m has T-multidesigns with leave P_2 and padding $P_2 + P_2$.

For each of the remaining triples $T = (G_1, G_2, H_1^3)$, we demonstrate a T-multidecomposition of K_8. We begin with the case $T = (H_i^8, H_j^4, H_1^3)$. Let $H \cong K_6$ be the graph induced by $\mathbb{Z}_6$, and factor it into T. Let $H' \cong G_{6,8}$ be the complement of H.

Lemma 2.6. For any $1 \le j \le 3$, if we remove the edges of H_j^4, from H, we can partition these edges and some of the edges of H' using only copies of H_1^3 to obtain (up to relabeling $V(H)$) the graph with the edge set $E(H) \cup \{\{6,7\}, \{0,6\}, \{1,6\}, \{4,7\}, \{5,7\}\}$.

Proof. Each H_j^4 has two connected components. After removing edges of H_j^4 from H, we get our first copy of H_1^3 from an edge of each component of H_j^4 and $\{6,7\}$. Two edges of H_j^4 remain. Our next copy of H_1^3 uses one of these edges. The other edges are formed by the vertices of the remaining edge of H_j^4 and 6 and 7, respectively, unless there is only one additional vertex available on the remaining edge. In this case, we choose the second vertex of the edge from one of the other vertices in H. There are now two vertices in H whose edges with 6 and 7 have not been used, and that are not on the remaining edge of H_j^4. Our last copy of H_1^3 is formed from the edges formed by these two vertices with 6 and 7, respectively, and the remaining edge of H_j^4. This completes the proof. $\square$

We then get a T-multidecomposition for all graph-triples with $j \neq 2$.

Corollary 2.7. Any graph-triple $T = (H_i^8, H_j^4, H_1^3)$ divides K_8 for $j = 1, 3$.

Proof. Fill in edges of K_8 as in Lemma 2.6. We partition the remaining edges with either $H_1^4 \cong [2,6,3,0,7,1]$, $[2,7,3,4,6,5]$ or $H_3^4 \cong [2,6,3,0,7,4]$, $[1,7,2,5,6,3]$. $\square$

We turn our attention to H_2^4.

Lemma 2.8. Given any factorization of the graph H into T, and any $i, j \in \mathbb{Z}_6$, we can remove the edges of H_1^3 from H and then add two copies of H_1^3 to achieve the graph with edge set $E(H) \cup \{67, i6, j7\}$.

Proof. Without loss of generality, let $H_1^3 \cong [0,1,2,3,4,5]$. If $ij \notin E(H_1^3)$, we can assume $i = 0$, $j = 5$ and add in $H_1^3 \cong [0,1,4,5,6,7]$, $[2,3,0,6,5,7]$. If $ij \in E(H_1^3)$, we can assume $i = 0$, $j = 1$ and add in $H_1^3 \cong [0,6,1,7,2,3]$, $[0,1,6,7,4,5]$. Each gives us the desired graph. $\square$

Corollary 2.9. (H_i^8, H_2^4, H_1^3) divides K_8.

Proof. Relabel $V(H)$ so that $H_2^4 \cong [0,1,2,3,4,5]$, and remove these vertices. We remove and insert edges as in Lemma 2.8 with $i = 0$, $j = 5$. We add in $H_1^3 \cong [2,3,1,6,4,7]$, $[1,2,0,7,4,6]$, and the remaining edges are $H_2^4 \cong [5,6,3,7,0,1]$, $[1,7,2,6,4,5]$. $\square$

Now we consider T-multidecompositions of K_8 for graph-triples of the form $T = (H_i^7, H_j^5, H_k^3)$. As before, we take an induced $H \cong K_6$ with $V(H) = \mathbb{Z}_6$ in K_8 and factor it into T. Let $H' \cong G_{6,8}$ be the complement of H.

Consider $j = 1$. We remove and add in copies of H_1^3 as in Lemma 2.8 with $i = 0$, $j = 5$, and we relabel $V(H)$ so that $H_1^5 \cong [3,4,5,0,1,2]$. Remove these edges, and add in $H_1^5 \cong [6,4,5,0,7,2]$, $[7,1,0,5,6,3]$, $[3,4,7,6,1,2]$.

For $j = 2$, we remove $H_1^3 \cong [0,1,2,3,4,5]$ from H and then add in $H_1^3 \cong [0,1,2,3,6,7]$, $[1,7,2,6,4,5]$. We partition the remaining edges with $H_2^5 \cong [7,5,0,1,6,4]$, $[6,5,0,2,7,3]$.

For $j = 3$, we remove $H_3^5 \cong [0,1,2,3,5,4]$. We then add in $H_1^3 \cong [1,4,2,3,6,7]$ and $H_3^5 \cong [0,7,2,1,6,3]$, $[2,6,1,0,7,5]$, $[5,4,6,0,1,7]$.

Next, consider $j = 4$. We remove $H_4^5 \cong [0, 1, 4, 3, 5, 2]$, and add in $H_1^3 \cong [1, 2, 3, 4, 5, 6]$, along with $H_4^5 \cong [0, 6, 7, 3, 5, 2]$, $[4, 6, 1, 0, 7, 3]$, $[5, 4, 7, 0, 2, 1]$.

For $j = 5$, we remove $H_5^5 \cong [0, 1, 3, 4, 5, 2]$. Without loss of generality, the copy of H_1^3 in the factorization of H is $[0, 3, 1, 4, 2, 5]$. We remove these edges and add in $H_1^3 \cong [7, 0, 6, 5, 3, 4]$, $[6, 4, 7, 1, 0, 3]$. We then add in $H_5^5 \cong [6, 7, 0, 1, 2, 3]$, $[6, 0, 7, 4, 5, 2]$, $[7, 5, 6, 1, 4, 2]$.

We next consider $j = 6$. We remove $H_6^5 \cong [0, 1, 4, 5, 2, 3]$ from the factorization of H. We add in $H_1^3 \cong [0, 3, 1, 2, 6, 7]$ and $H_6^5 \cong [0, 6, 4, 5, 1, 7]$, $[6, 4, 2, 3, 7, 5]$, $[6, 2, 1, 0, 7, 3]$.

Finally, consider $j = 7$. We remove $H_7^5 \cong [0, 1, 5, 4, 3, 2]$ from the factorization of H. We add in $H_1^3 \cong [6, 4, 7, 0, 2, 3]$ and $H_7^5 \cong [0, 1, 7, 3, 2, 6]$, $[1, 2, 3, 6, 4, 7]$, $[6, 7, 0, 2, 4, 5]$.

We summarize our results as follows.

Theorem 2.10. Let $T = (G_1, G_2, H_1^3)$ be a graph-triple of order 6, and let $m \geq 6$.

1. If $m \neq 7$, and if either $m \equiv 0, 1 \pmod 3$ or $(G_1, G_2) \neq (H_i^6, H_j^6)$ for all $1 \leq i, j \leq 11$, then T divides K_m.

2. T divides K_7 if and only if either $(G_1, G_2) = (H_i^7, H_j^5)$ with $(i, j) \in \{(4, 2), (5, 2), (5, 3), (5, 7), (6, 2), (8, 3)\}$ or $(G_1, G_2) = (H_i^6, H_j^6)$ with $(i, j) \neq (1, 8)$.

3. Let $T = (H_i^6, H_j^6, H_1^3)$ with $m \equiv 2 \pmod 3$. K_m has T-multidesigns with leave P_2 and padding $P_2 + P_2$.

4. If $T = (H_1^6, H_8^6, H_1^3)$, then there exist T-multidesigns of K_7 whose leave and padding are both P_4.

5. If $T = (H_i^8, H_j^4, H_1^3)$, then there exist T-multidesigns of K_7 with leave $P_2 + P_2$ and padding P_2.

6. If $T = (H_i^7, H_j^5, H_1^3)$, then there exist T-multipackings of K_7 with leave P_2 for all $(i, j) \neq (3, 6)$. If $(i, j) = (3, 6)$, we have an optimal leave of $P_3 + P_2$.

7. If $T = (H_i^7, H_j^5, H_1^3)$, then there exist T-multicoverings of K_7 with leave P_2 for $(i, j) \neq (10, 1)$. For $(i, j) = (10, 1)$, we get an optimal leave of P_3.

3 Multidesigns for Graph-Triples (H_i^5, H_j^5, H_k^5)

We now consider graph-triples of order 6 of the form $T = (H_i^5, H_j^5, H_k^5)$. We construct multidesigns recursively as before, but we address T- multidecompositions separately. Note that K_m has $\frac{m(m-1)}{2}$ edges, so a necessary condition for a T-multidecomposition is $m \equiv 0, 1 \pmod 5$. The following gives us our induction step and delineates the necessary base cases.

Lemma 3.1. Suppose that $T = (H_i^5, H_j^5, H_k^5)$ is a graph-triple of order 6. Then

1. If T divides K_{10}, and if the edges of $K_{5,5}$, $G_{5,10}$, and $G_{6,11}$ can be partitioned into copies of H_i^5, H_j^5, and H_k^5, then T divides all K_m for $m \geq 6$ and $m \equiv 0{,}1 \pmod 5$.

2. If T divides K_{10} and K_{11}, and if the edges of $K_{5,5}$, $K_{2,5}$, and $G_{6,n}$ can be partitioned into copies of H_i^5, H_j^5, and H_k^5 for $n = 10$ or 11, then T divides all K_m for $n \geq 6$ and $m \equiv 0 \pmod 5$.

3. If T divides K_{11} and K_{16}, and K_{5k} for all $k \geq 2$, and if the edges of $K_{5,5}$ and $K_{2,5}$ can be partitioned into copies of H_i^5, H_j^5, and H_k^5, then T divides all K_m for $m \geq 6$ and $m \equiv 1 \pmod 5$.

4. If T divides K_{10}, and if the edges of $K_{2,5}$ can be partitioned into copies of H_i^5, H_j^5, and H_k^5, then T divides K_{16}

Proof. For (1), we start with $m \equiv 0 \pmod 5$. Let $m = 5k$. By assumption, T divides K_{10}, which gives us the case $k = 2$. For $k \geq 3$, partition the vertices of K_m into sets A, B, and C of sizes 5, 5, and $5k - 10$, respectively. By induction, T divides K_{5k-5}, which is isomorphic to the graph induced by $B \cup C$. The remaining edges among $A \cup B$ form $G_{5,10}$, and the remaining edges between B and C can be partitioned into $k - 2$ copies of $K_{5,5}$. For $m \equiv 1 \pmod 5$, let $m = 5k + 1$. A factorization of K_6 gives us $k = 1$. For $k \geq 2$, partition the vertices of K_m into sets A, B, and C of sizes 5, 6, and $5k - 10$, respectively. By induction, T divides K_{5k-4}, which is isomorphic to the graph induced by $B \cup C$. The remaining edges of $A \cup B$ form $G_{6,11}$, and the remaining edges between A and C form $k - 2$ copies of $K_{5,5}$. This gives us (1).

For (2), we first show that T divides K_{15}. Partition the vertices of K_{15} into sets A, B, and C of size 4, 5, and 6, respectively. Now T divides the graph induced by $B \cup C$. The remaining edges among $A \cup C$ form a copy of $G_{6,10}$ and the remaining edges among $A \cup B$ form two copies of $K_{2,5}$, which gives us the case $n = 10$. The case $n = 11$ is similar. For $m = 5k$, $k \geq 4$, partition the vertices of K_{5k} into sets A and B of size 10 and $5k - 10$, respectively. We have that T divides the graphs induced by A and B. The remaining edges form $2k - 4$ copies of $K_{5,5}$.

For (3), we need only show the result for $m = 5k + 1$, $k \geq 4$. We partition the vertices of K_{5k+1} into sets A, B, and C of sets 5, 6, and $5k - 10$, respectively. By induction, T divides the graph induced by $A \cup B$, and T divides the graph induced by C by assumption. The remaining edges among $A \cup C$ form $k - 2$ copies of $K_{5,5}$, and the remaining edges among $B \cup C$ form $3k - 6$ copies of $K_{2,5}$, which completes the proof.

For (4), we partition the vertices of K_{16} into sets A and B of size 6 and 10, respectively. We factor the graph induced by A into T. Also, T divides the graph induced by B. The remaining edges form six copies of $K_{2,5}$, which completes the proof. $\square$

For the base cases, we first consider triples with H^5_1. By Lemmas 1.1 and 3.1(1), we require only the following lemma.

Lemma 3.2. H^5_1 divides $K_{5,5}$, $G_{5,10}$, $G_{6,10}$, and $G_{6,11}$.

Proof. For $K_{5,5}$, let the partite sets be given by $\mathbb{Z}_5$ and $\{a, b, c, d, e\}$. An H^5_1-decomposition is $[b, 2, c, 0, a, 1]$, $[e, 4, a, 1, d, 3]$, $[c, 3, b, 2, d, 4]$, $[e, 0, d, 3, a, 2]$, $[c, 1, e, 4, b, 0]$.

For $G_{5,10}$, an H^5_1-decomposition is $[7, 1, 8, 0, 5, 6]$, $[5, 2, 8, 0, 6, 9]$, $[5, 3, 8, 0, 9, 7]$, $[7, 4, 9, 1, 5, 8]$, $[6, 3, 7, 2, 9, 8]$, $[6, 4, 5, 3, 9, 1]$, $[7, 2, 6, 4, 8, 0]$.

An H^5_1-decomposition of $G_{6,10}$ is $[1, 8, 9, 6, 0, 7]$, $[7, 5, 8, 6, 1, 9]$, $[8, 2, 7, 6, 9, 3]$, $[9, 4, 8, 7, 6, 5]$, $[8, 0, 9, 7, 3, 6]$, $[6, 2, 9, 8, 7, 4]$.

For $G_{6,11}$, an H^5_1-decomposition is $[8, 9, 1, 0, 6, 7]$, $[6, 8, 4, 5, 10, 9]$, $[10, 6, 3, 2, 9, 7]$, $[4, 7, 2, 0, 8, 10]$, $[8, 5, 9, 0, 7, 1]$, $[10, 3, 9, 4, 6, 2]$, $[7, 5, 6, 2, 8, 3]$, $[10, 1, 6, 4, 9, 0]$. $\square$

Now we move on to the case $i = 2$. Lemma 1.1 and Lemma 3.1(2), (3), and (4) reduce our problem to the following two lemmas:

Lemma 3.3. There exists an H^5_2-decomposition of $K_{5,5}$, $K_{2,5}$, and $G_{6,10}$.

Proof. For $K_{5,5}$, let the partite sets be $\mathbb{Z}_5$ and $\{a, b, c, d, e\}$. An H^5_2-decomposition is $[1, d, a, b, 0, c]$, $[0, d, a, b, 1, e]$, $[3, d, a, b, 2, c]$, $[4, d, a, e, 3, b]$, $[2, d, a, c, 4, e]$.

For $K_{2,5}$, we have partite sets $\{a, b\}$ and $\mathbb{Z}_5$. An H^5_2-decomposition is $[b, 3, 0, 1, a, 2]$, $[a, 3, 0, 1, b, 4]$.

Finally, for $G_{6,10}$, an H^5_2-decomposition is $[7, 1, 8, 9, 6, 0]$, $[6, 3, 7, 9, 8, 2]$, $[2, 7, 6, 8, 1, 9]$, $[4, 6, 7, 8, 3, 9]$, $[0, 8, 6, 7, 5, 9]$, $[8, 5, 6, 9, 7, 4]$. $\square$

Lemma 3.4. There exists a T-multidecomposition of K_{11}.

Proof. We factor the K_6 induced by $\mathbb{Z}_6$ into T so that $H^5_2 \cong [4, 5, 2, 3, 0, 1]$. Remove $\{0, 2\}$ and have it reappear as $\{0, 10\}$. This will still be a copy of H^5_2. We partition the remaining edges into $H^5_2 \cong [7, 1, 8, 9, 6, 0]$, $[9, 3, 8, 6, 7, 2]$, $[8, 1, 6, 7, 10, 3]$, $[10, 5, 7, 8, 9, 4]$, $[6, 3, 8, 9, 10, 2]$, $[7, 3, 0, 2, 8, 5]$, $[5, 9, 7, 8, 4, 6]$, $[0, 2, 6, 10, 1, 9]$ $\square$

What remains is the triple (H^5_3, H^5_4, H^5_7). Lemmas 1.1 and 3.1(1) reduce our problem to the following.

Lemma 3.5. The following are true:

1. There exist an (H_3^5, H_4^5)-multidecompositions of $K_{5,5}$ and $G_{5,10}$.
2. There exists an (H_3^5, H_4^5, H_7^5)-multidecompisiton of $G_{6,10}$.
3. There exists an H_4^5-decomposition of $G_{6,11}$.

Proof. For (1),we start with $K_{5,5}$. As before, let the partite sets be $\mathbb{Z}_5$ and $\{a, b, c, d, e\}$. An (H_3^5, H_4^5)-multidecomposition is $H_3^5 \cong [d, 1, e, 0, 4, b]$, $[b, 3, a, 4, 0, d]$ and $H_4^5 \cong [2, a, 0, b, c, 1]$, $[2, e, 4, c, d, 3]$, $[1, c, 2, b, d, 3]$.

For $G_{5,10}$, an (H_3^5, H_4^5)-multidecomposition is $H_3^5 \cong [1, 9, 5, 3, 8, 2]$ and $H_4^5 \cong [0, 5, 6, 2, 3, 1]$, $[9, 8, 7, 2, 3, 4]$, $[4, 9, 6, 0, 1, 3]$, $[2, 5, 8, 1, 3, 4]$, $[8, 0, 7, 5, 4, 9]$, $[4, 6, 7, 1, 9, 8]$.

For (2), an (H_3^5, H_4^5, H_7^5)-multidecomposition of $G_{6,10}$ is $H_3^5 \cong [2, 6, 0, 7, 8, 1]$, $[3, 6, 5, 7, 8, 4]$, $[8, 2, 7, 6, 1, 9]$, $[6, 8, 3, 7, 9, 0]$, $H_4^5 \cong [1, 7, 9, 4, 3, 8]$, and $H_7^5 \cong [5, 8, 7, 4, 6, 9]$.

Finally, for (3), an H_4^5-decomposition of $G_{6,11}$ is $[0, 6, 7, 2, 5, 1]$, $[2, 8, 9, 4, 5, 1]$, $[0, 7, 10, 4, 5, 1]$, $[2, 6, 8, 5, 3, 4]$, $[0, 8, 10, 1, 6, 4]$, $[0, 9, 7, 8, 4, 1]$, $[3, 6, 9, 2, 10, 5]$, $[0, 10, 3, 7, 9, 2]$. □

Putting Lemma 3.1, Lemma 3.2, Lemma 3.3, and Lemma 3.5 together, we get the following.

Theorem 3.6. For each $m \geq 6$ with $m \equiv 0, 1 \pmod 5$, any triple $T = (H_i^5, H_j^5, H_k^5)$ divides K_m.

We now turn to multidesigns for the cases $m \equiv 2, 3, 4 \pmod 5$. If $m \equiv 2, 4 \pmod 5$, then the number of edges of K_m is congruent 1 mod 5, and so an optimal multidesign must have at least a 1-edge leave or 4-edge padding. If $m \equiv 3 \pmod 5$, the number of edges is congruent 3 mod 5, and so an optimal multidesign must have at least a 3-edge leave or 2-edge padding. We show that each of these lower bounds is achieved for all triples.

We begin with some designs that will prove useful to us.

Lemma 3.7.

1. There are H_i^5-packings of $K_{6,6}$ with leave P_2 for $i = 1, 2, 3, 4$.
2. H_i^5 divides $K_{4,5}$ for $i = 1, 3$.
3. H_3^5 divides $K_{3,5}$ and H_1^5 divides $K_{5,7}$.

Proof. Let $\mathbb{Z}_6$ and $\{a, b, c, d, e, f\}$ be the partite sets of $K_{6,6}$. For (1), an H_1^5-packing is $[2, a, 1, c, 3, b]$, $[5, d, 4, f, 0, e]$, $[4, c, 2, d, 1, f]$, $[5, b, 1, d, 0, a]$, $[3, d, 2, e, 4, a]$, $[0, b, 4, e, 1, c]$, $[f, 5, c, 2, e, 3]$, with leave $\{2, f\}$. An H_2^5-packing is $[b, 3, 0, 1, a, 2]$, $[4, c, f, e, 5, d]$, $[a, 5, 1, 0, b, 4]$, $[5, b, d, e, 2, c]$, $[c, 1, 2, 4, f, 3]$, $[1, d, c, e, 0, f]$, $[d, 0, 1, 4, e, 3]$ with leave $\{3, a\}$. An H_3^5-packing is $[3, a, 1, b, c, 2]$, $[0, a, 4, e, f, 5]$, $[4, d, 3, c, f, 0]$, $[4, b, 2, e, c, 5]$, $[0, c, 1, d, f, 4]$, $[3, e, 0, b, d, 5]$, $[3, f, 1, e, d, 2]$ with leave $\{3, b\}$. Finally, an H_4^5-packing is $[b, 0, a, 1, 2, c]$, $[d, 1, b, 2, 3, c]$, $[d, 2, c, 3, 4, e]$, $[f, 3, d, 5, 4, e]$, $[b, 5, f, 1, 2, c]$, $[d, 0, e, 1, 5, f]$, $[e, 4, a, 3, 5, f]$ with leave $\{4, b\}$.

For (2), let the partite sets of $K_{4,5}$ be $\mathbb{Z}_5$ and $\{a, b, c, d\}$. An H_1^5-decomposition is $[2, a, 1, c, 3, b]$, $[d, 4, a, 1, c, 0]$, $[a, 0, b, 2, d, 3]$, $[b, 1, d, 2, c, 4]$. An H_3^5-decomposition is $[3, a, 1, b, c, 2]$, $[3, b, 2, d, a, 4]$, $[4, c, 3, d, a, 0]$, $[4, d, 1, c, b, 0]$.

For (3), label the vertices of $K_{3,5}$ and $K_{5,7}$ similarly as before. An H_3^5-decomposition of $K_{3,5}$ is $[a, 1, b, 2, 3, c]$, $[c, 4, a, 2, 3, b]$, $[b, 0, a, 3, 2, c]$. An H_1^5-decomposition of $K_{5,7}$ is $[2, a, 1, c, 3, b]$, $[1, c, 2, e, 4, d]$, $[5, e, 3, a, 6, c]$, $[0, a, 4, b, 5, d]$, $[3, a, 5, e, 6, d]$, $[4, b, 6, e, 0, c]$, $[1, b, 0, d, 2, e]$. □

Lemma 3.8. Let $T = (H_i^5, H_j^5, H_k^5)$. Suppose that K_8 has a T-multipacking with leave L, and that K_m has a T-multipacking with leave P_2 for $m = 7, 9, 12, 14$.

1. If $m \equiv 2, 4 \pmod 5$ and $m \geq 17$, then K_m has T-multipacking with leave P_2.
2. If $m \equiv 3 \pmod 5$ and $m \geq 18$, then K_m has a T-multipacking with leave L.

Proof. Note that T includes either H_1^5, H_2^5, or H_3^5. Suppose T includes H_1^5. For (1), we begin with $m \equiv 2 \pmod 5$, so $m = 5k + 2$, $k \geq 3$. We first partition the vertices of K_m into sets A and B of size 7 and $5k - 5$, respectively. Now B induces a K_{5k-5}, which T divides by Theorem 3.6. The graph induced by A has a multipacking with leave P_2 by assumption, and the remaining edges form copies of $K_{5,7}$, which T divides by Lemma 3.7(3). For H_2^5, we partition the vertices of K_m into sets A, B, and C of size 6, 6, and $5k - 10$, respectively. By Theorem 3.6, T divides the graph induced by $B \cup C$, and the graph induced by A can be factored into T. The remaining edges among $A \cup C$ form copies of $K_{2,5}$, which T divides by Lemma 3.3. The remaining edges among $A \cup B$ form $K_{6,6}$, which has a T-multipacking with leave P_2 by Lemma 3.7(1). The argument for H_3^5 is almost identical, using Lemma 3.7(3) in place of Lemma 3.3.

Now let $m \equiv 4 \pmod 5$, so $m = 5k + 4$, $k \geq 3$. We begin with H_1^5. Partition the vertices of K_m into sets A, B, and C of size 4, 5, and $5k-5$, respectively. Now $A \cup B$ induces a K_9, which has a T-multipacking with leave P_2 by assumption. The set C induces a K_{5k-5}, which T divides. The remaining edges among $A \cup C$ form copies of $K_{4,5}$, which H_1^5 divides by Lemma 3.7(2). The remaining edges among $B \cup C$ form copies of $K_{5,5}$, which H_1^5 divides by Lemma 3.2. The argument for H_2^5 is identical, except we partition the copies of $K_{4,5}$ into copies of $K_{2,5}$ and use Lemma 3.3. For H_3^5, we partition the vertices of K_m into sets A and B of size 9 and $5k - 5$, respectively. The set A induces a K_9, which has a T-multipacking with leave P_2 by assumption. Moreover, T divides the graph induced by B, which is K_{5k-5}. The remaining edges form copies of $K_{3,5}$, which H_3^5 divides by Lemma 3.7(3).

For (2), we have $m = 5k + 3$, $k \geq 3$, and we partition the vertices of K_m into sets A and B of size 8 and $5k - 5$, respectively. We have a T-multidecomposition of the subgraph induced by B as well as a T-multipacking of the graph induced by A with leave L. The remaining vertices can be partitioned into either copies of $K_{2,5}$ or $K_{4,5}$. H_2^5 divides the first of these, and H_1^5 and H_3^5 divide the second. □

This reduces the multipacking problem to finding optimal multipackings for K_m, $m = 7, 8, 9, 12, 13, 14$. We construct these multipackings so that the leave is a subgraph of one of the graphs in the triple, which yields an optimal T-multicovering. We begin with a technical lemma.

Lemma 3.9. Let G be the graph given by $K_{3,3}$ along with an additional 2-path among the vertices of one of the partite sets P. Then G has an H_1^5-packing with leave P_2. Furthermore, the leave is between two vertices in P.

Proof. Let the partite sets of $K_{3,3}$ be $\mathbb{Z}_3$ and $\{a, b, c\}$, and let the additional edges of G be $\{0, 1\}$ and $\{1, 2\}$. We then have the H_1^5-packing $[b, 2, c, 0, a, 1]$, $[1, 2, a, b, 0, c]$ with leave $\{0, 1\}$. □

The following gives us optimal multipackings for $m \equiv 2, 4 \pmod 5$.

Lemma 3.10. For $m = 7, 9, 12, 14$ and $T = (H_i^5, H_j^5, H_k^5)$, K_m has a T-multipacking with leave P_2.

Proof. For K_7, we begin with triples T that include H_2^5. Now $\mathbb{Z}_6$ induces a K_6, which we can factor into T. We remove the copy of H_2^5 (say $[1, 0, 4, 5, 3, 2]$). We then insert $H_2^5 \cong [1, 0, 5, 3, 6, 2]$, $[3, 5, 1, 0, 6, 4]$, which gives us a multipacking with leave $\{2, 3\}$. For triples that include H_3^5, we factor an induced K_6 into T and remove $H_3^5 \cong [0, 1, 2, 3, 5, 4]$. We then insert $H_3^5 \cong [3, 6, 0, 1, 5, 4], [5, 6, 2, 3, 4, 1]$. The leave is $\{1, 2\}$. The remaining triple is (H_1^5, H_5^5, H_7^5). We remove $H_1^5 \cong [3, 4, 5, 0, 1, 2]$ and insert $H_1^5 \cong [6, 2, 1, 3, 4, 5]$ and $H_7^5 \cong [0, 1, 2, 3, 4, 6]$. The leave is $\{3, 6\}$.

For multipackings of K_9, we first consider triples that include either H_1^5 or H_3^5. By Theorem 1.1, it suffices to construct an H_i^5-decomposition of $G_{6,9}$ for $i = 1, 3$. An H_1^5-packing is $[6, 8, 4, 3, 7, 2]$, $[1, 8, 2, 0, 6, 7]$, $[7, 8, 5, 3, 6, 4]$, $[7, 0, 8, 1, 6, 5]$ with leave $\{3, 8\}$. An H_3^5-packing is $[0, 6, 1, 8, 2, 7]$, $[3, 7, 0, 8, 6, 5]$, $[2, 8, 4, 7, 6, 3]$, $[5, 8, 6, 2, 1, 7]$, with leave $\{4, 6\}$.

Two triples remain, both of which include H_2^5. We factor the K_6 induced by $\mathbb{Z}_6$ into $H_2^5 \cong [1, 0, 4, 5, 3, 2]$. Remove the edges of this subgraph, and insert $H_2^5 \cong [1, 0, 8, 2, 7, 6]$, $[3, 7, 1, 2, 8, 4]$, $[2, 1, 4, 5, 6, 3]$, $[3, 5, 0, 2, 6, 8]$, $[8, 5, 1, 4, 7, 0]$. The leave is $\{5, 7\}$.

For multipackings of K_{12}, we first consider triples T that include either H_2^5 or H_4^5. Partition the vertices of K_{12} into two sets of size 6. Each subset induces a K_6, which can be factored into T. The remaining vertices form $K_{6,6}$, which has a T-multipacking with leave P_2 by Lemma 3.7(1).

The only remaining triples include a copy of H_1^5. Partition the vertices of K_{12} into the sets $A = \mathbb{Z}_6$ and $B = \mathbb{Z}_{12} - \mathbb{Z}_6$, and factor each induced subgraph into T. We remove $H_1^5 \cong [3, 4, 5, 0, 1, 2]$. The remaining edges among $\{0, 1, 2, 6, 7, 8\}$ form the graph G from Lemma 3.9. We execute an H_1^5-packing with leave $\{0, 1\}$. We do the same thing with the vertices $\{0, 1, 2, 9, 10, 11\}$, only this time with a leave of $\{1, 2\}$. The same process with the vertices in $\{3, 4, 5\} \cup B$ give us a T-multipacking with leave $\{2, 3\}$.

For multipackings of K_{14}, T divides the subgraph induced by $\mathbb{Z}_{11}$ by Theorem 3.6. For triples that include H_1^5, we remove $H_1^5 \cong [3, 4, 5, 0, 1, 2]$ from the T-decomposition. We then insert $H_1^5 \cong [11, 1, 2, 13, 12, 0]$, $[6, 11, 10, 1, 0, 13]$, $[7, 11, 8, 12, 1, 13]$, $[12, 4, 3, 13, 11, 5]$, $[10, 12, 6, 4, 5, 13]$, $[8, 12, 7, 11, 4, 13]$, $[3, 11, 9, 12, 2, 13]$, $[12, 9, 13, 3, 2, 11]$. The leave is $\{3, 12\}$. We proceed similarly for H_2^5, removing $H_2^5 \cong [1, 0, 4, 5, 3, 2]$ and adding in $H_2^5 \cong [13, 7, 0, 1, 12, 2]$, $[13, 4, 7, 1, 11, 5]$, $[5, 3, 13, 11, 6, 12]$, $[13, 0, 7, 3, 12, 10]$, $[1, 2, 12, 10, 11, 0]$, $[3, 4, 12, 13, 8, 11]$, $[4, 11, 1, 3, 13, 12]$, $[2, 3, 12, 13, 9, 11]$. The leave is $\{11, 13\}$. Our last case is the triple (H_3^5, H_4^5, H_7^5). We take a T-multidecomposition on the subgraph induced by $\mathbb{Z}_{10}$. We add in $H_4^5 \cong [1, 10, 11, 2, 3, 0]$, $[0, 11, 12, 4, 5, 1]$, $[2, 12, 13, 5, 4, 3]$, $[3, 10, 12, 0, 1, 2]$, $[4, 11, 13, 2, 3, 5]$, $[5, 10, 13, 0, 1, 4]$. Then add in $H_3^5 \cong [6, 10, 8, 12, 11, 7]$, $[13, 9, 12, 7, 6, 11]$, $[7, 13, 8, 11, 12, 6]$. The leave is $\{9, 10\}$. □

We now proceed to optimal multipackings for $m = 8, 13$.

Lemma 3.11. For $m = 8, 13$ and $T = (H_i^5, H_j^5, H_k^5)$, K_m has a multipacking with a 3-edge leave that is a subgraph of least one of the graphs in T.

Proof. We begin with K_8. For multipackings into triples T that include H_1^5, we factor an induced K_6 into T and remove $H_1^5 \cong [3, 4, 5, 0, 1, 2]$. We then add in $H_1^5 \cong [3, 7, 5, 0, 6, 2]$, $[7, 4, 5, 6, 1, 0]$, $[6, 3, 4, 1, 2, 7]$. The leave is $\{4, 6\}$, $\{5, 6\}$, $\{1, 7\}$, which is a subgraph of H_1^5. For triples that include H_2^5, we factor a K_6 into T as before. We then add in $H_2^5 \cong [7, 3, 0, 1, 6, 2]$, $[6, 5, 0, 1, 7, 4]$. The leave is $\{3, 6\}$, $\{6, 7\}$, $\{5, 7\}$, which is a subgraph of H_2^5. What remains is the triple (H_3^5, H_4^5, H_7^5). We remove $H_4^5 \cong [1, 2, 3, 4, 5, 0]$

of from a factorization of K_6 into T and add in $H_4^5 \cong [1,6,7,4,5,0]$, $[1,2,6,3,4,7]$, $[0,7,3,4,5,1]$. The leave is $\{0,2\}$, $\{2,3\}$, $\{5,6\}$, which is a subgraph of any graph in T.

For K_{13}, we begin with triples that include H_1^5. By Theorem 3.6, T divides the graph induced by $\mathbb{Z}_{11}$. We remove $H_1^5 \cong [3,4,5,0,1,2]$ and insert $H_1^5 \cong [12,2,1,0,11,3]$, $[12,0,1,6,11,7]$, $[11,2,3,6,12,1]$, $[5,12,8,3,4,11]$, $[9,11,10,5,4,12]$. The leave is $\{8,11\}$, $\{11,12\}$, $\{12,10\}$, which is a subgraph of H_1^5. For triples that include H_2^5, we get a T-multidecomposition of the subgraph induced by $\mathbb{Z}_{11}$ by Theorem 3.6. The bipartite subgraph induced by $\mathbb{Z}_5$ and $\{12,13\}$ is isomorphic to $K_{2,5}$, which H_2^5 divides by Lemma 3.3. We add in $H_2^5 \cong [11,5,7,8,12,6]$, $[11,7,5,10,12,9]$. The leave is $\{8,11\}$, $\{10,11\}$, $\{11,12\}$, which is a subgraph of H_2^5. The final triple is $T = (H_3^5, H_4^5, H_7^5)$, which divides the subgraph induced by $\mathbb{Z}_{10}$. The remaining edges minus the subgraph induced by $\{10,11,12\}$ form two copies of $K_{3,5}$, which can be partitioned into copies of H_3^5 by Lemma 3.7(3). The leave is $\{10,11\}$, $\{11,12\}$, $\{12,10\}$, which is a subgraph of H_7^5. □

The leaves in the multipackings of Lemmas 3.10 and 3.11 are subgraphs of at least one graph in the given graph-triple. Thus, if the leave has size s, we can obtain a multicovering of size $5-s$. We summarize this, along with the other results of this section, in the following theorem.

Theorem 3.12. Let $T = (H_i^5, H_j^5, H_k^5)$ be a graph-triple of order 6, and let $m \geq 6$.

1. If $m \equiv 0, 1 \pmod 5$, then T divides K_m.
2. If $m \equiv 2$ or $4 \pmod 5$, then K_m has a T-multipacking with leave P_2 and a T-multicovering with a 4-edge padding.
3. If $m \equiv 3 \pmod 5$, then K_m has a T-multipacking with a leave of three edges and a T-multicovering with a 2-edge padding.

4 Conclusion

We have settled the T-multidesign problem of K_m into graph-triples T of order 6 that are of the form (G_1, G_2, H_1^3) or (H_i^5, H_j^5, H_k^5), but the problem is still open for graph-triples of the forms (H_i^7, H_j^4, H_k^4) and (H_i^6, H_j^5, H_k^4). Another extension of this work will be to investigate multidesigns into graph-triples of order 6 with various specified leaves.

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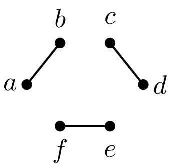
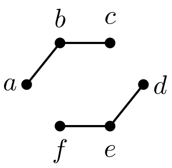
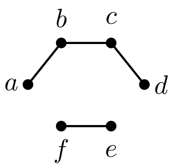
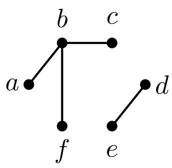
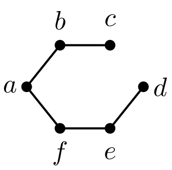
A Graphs of Order 6 that are Part of Graph-Triples

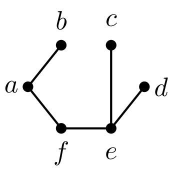
$H_1^3 \cong$ 

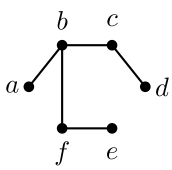
$H_1^4 \cong$ 

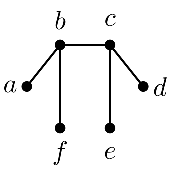
$H_2^4 \cong$ 

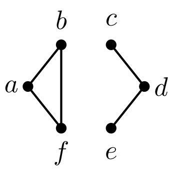
$H_3^4 \cong$ 

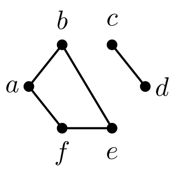
$H_1^5 \cong$ 

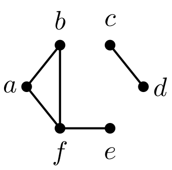
$H_2^5 \cong$ 

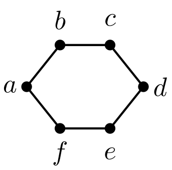
$H_3^5 \cong$ 

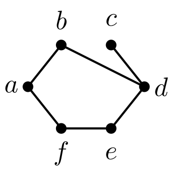
$H_4^5 \cong$ 

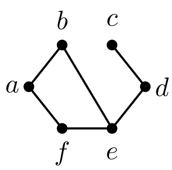
$H_5^5 \cong$ 

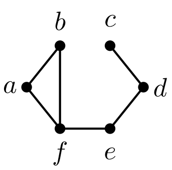
$H_6^5 \cong$ 

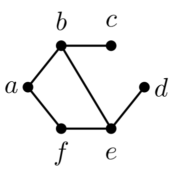
$H_7^5 \cong$ 

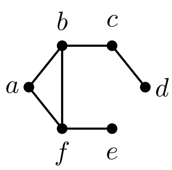
$H_1^6 \cong$ 

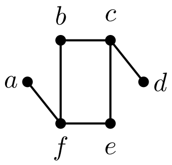
$H_2^6 \cong$ 

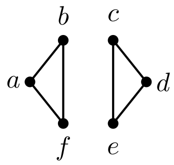
$H_3^6 \cong$ 

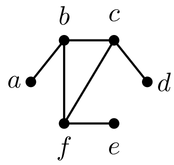
$H_4^6 \cong$ 

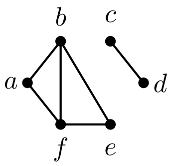
$H_5^6 \cong$ 

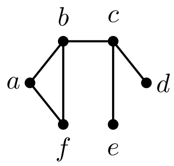
$H_6^6 \cong$ 

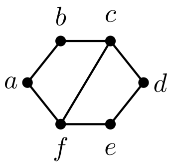
$H_7^6 \cong$ 

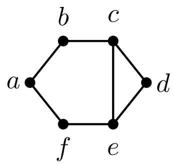
$H_8^6 \cong$ 

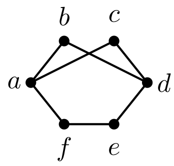
$H_9^6 \cong$ 

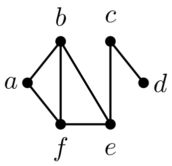
$H_{10}^6 \cong$ 

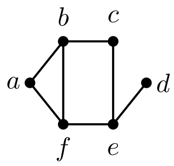
$H_{11}^6 \cong$ 

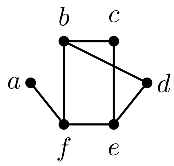
$H_1^7 \cong$ 

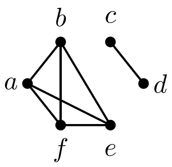
$H_2^7 \cong$ 

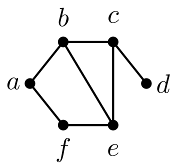
$H_3^7 \cong$ 

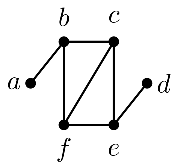
$H_4^7 \cong$ 

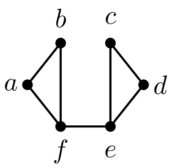
$H_5^7 \cong$ 

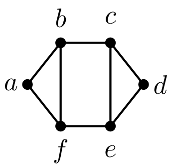
$H_6^7 \cong$ 

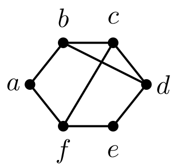
$H_7^7 \cong$ 

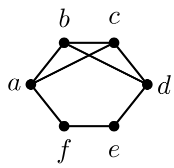
$H_8^7 \cong$ 

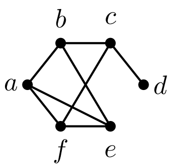
$H_9^7 \cong$ 

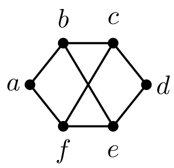
$H_{10}^7 \cong$ 

$H_1^8 \cong$ 

$H_2^8 \cong$ 

$H_3^8 \cong$ 

$H_4^8 \cong$ 

$H_5^8 \cong$ 

B The Graph-Triples of Order 6

The graph triples of order six $T = (G_1, G_2, G_3) = (H_{i_1}^{j_1}, H_{i_2}^{j_2}, H_{i_3}^{j_3})$, where j_k represents the number of edges in the graph G_k.

For $j_1 = 8,\ j_2 = 4,\ j_3 = 3$,

$$
\begin{array}{rlll}
T = (G_1, G_2, G_3) \in & \{(H_1^8,\ H_1^4,\ H_1^3), & (H_1^8,\ H_2^4,\ H_1^3), & (H_2^8,\ H_2^4,\ H_1^3), \\
& (H_3^8,\ H_1^4,\ H_1^3), & (H_4^8,\ H_3^4,\ H_1^3)\}, & (H_5^8,\ H_1^4,\ H_1^3).
\end{array}
$$

For $j_1 = 7,\ j_2 = 4,\ j_3 = 4$,

$$
\begin{array}{rlll}
T = (G_1, G_2, G_3) \in & \{(H_1^7,\ H_1^4,\ H_2^4), & (H_2^7,\ H_1^4,\ H_2^4), & (H_3^7,\ H_1^4,\ H_2^4), \\
& (H_4^7,\ H_1^4,\ H_2^4), & (H_4^7,\ H_1^4,\ H_3^4), & (H_4^7,\ H_2^4,\ H_3^4), \\
& (H_5^7,\ H_1^4,\ H_2^4), & (H_5^7,\ H_2^4,\ H_3^4), & (H_6^7,\ H_1^4,\ H_2^4), \\
& (H_6^7,\ H_2^4,\ H_3^4), & (H_8^7,\ H_1^4,\ H_3^4), & (H_9^7,\ H_1^4,\ H_2^4), \\
& (H_{10}^7,\ H_1^4,\ H_2^4)\}. & &
\end{array}
$$

For $j_1 = 7,\ j_2 = 5,\ j_3 = 3$,

$$
\begin{array}{rlll}
T = (G_1, G_2, G_3) \in & \{(H_1^7,\ H_1^5,\ H_1^3), & (H_2^7,\ H_1^5,\ H_1^3), & (H_2^7,\ H_5^5,\ H_1^3), \\
& (H_3^7,\ H_1^5,\ H_1^3), & (H_3^7,\ H_6^5,\ H_1^3), & (H_4^7,\ H_2^5,\ H_1^3), \\
& (H_5^7,\ H_2^5,\ H_1^3), & (H_5^7,\ H_3^5,\ H_1^3), & (H_5^7,\ H_7^5,\ H_1^3), \\
& (H_6^7,\ H_2^5,\ H_1^3), & (H_8^7,\ H_3^5,\ H_1^3), & (H_9^7,\ H_4^5,\ H_1^3), \\
& (H_{10}^7,\ H_1^5,\ H_1^3)\}. & &
\end{array}
$$

For $j_1 = 6,\ j_2 = 5,\ j_3 = 4$,

$$
\begin{array}{rlll}
T = (G_1, G_2, G_3) \in & \{(H_1^6,\ H_1^5,\ H_1^4), & (H_1^6,\ H_1^5,\ H_2^4), & (H_1^6,\ H_5^5,\ H_2^4), \\
& (H_1^6,\ H_6^5,\ H_1^4), & (H_2^6,\ H_1^5,\ H_1^4), & (H_2^6,\ H_1^5,\ H_2^4), \\
& (H_2^6,\ H_1^5,\ H_3^4), & (H_2^6,\ H_2^5,\ H_2^4), & (H_2^6,\ H_3^5,\ H_1^4), \\
& (H_2^6,\ H_3^5,\ H_2^4), & (H_2^6,\ H_5^5,\ H_2^4), & (H_2^6,\ H_6^5,\ H_2^4), \\
& (H_2^6,\ H_7^5,\ H_1^4), & (H_2^6,\ H_7^5,\ H_2^4), & (H_3^6,\ H_1^5,\ H_1^4), \\
& (H_3^6,\ H_1^5,\ H_2^4), & (H_3^6,\ H_1^5,\ H_3^4), & (H_3^6,\ H_2^5,\ H_1^4), \\
& (H_3^6,\ H_2^5,\ H_2^4), & (H_3^6,\ H_3^5,\ H_2^4), & (H_3^6,\ H_5^5,\ H_1^4), \\
& (H_3^6,\ H_5^5,\ H_3^4), & (H_3^6,\ H_6^5,\ H_2^4), & (H_3^6,\ H_7^5,\ H_1^4), \\
& (H_4^6,\ H_1^5,\ H_1^4), & (H_4^6,\ H_1^5,\ H_2^4), & (H_4^6,\ H_1^5,\ H_3^4), \\
& (H_4^6,\ H_2^5,\ H_2^4), & (H_4^6,\ H_3^5,\ H_1^4), & (H_4^6,\ H_3^5,\ H_2^4), \\
& (H_4^6,\ H_5^5,\ H_2^4), & (H_4^6,\ H_6^5,\ H_1^4), & (H_4^6,\ H_6^5,\ H_3^4), \\
& (H_4^6,\ H_7^5,\ H_1^4), & (H_4^6,\ H_7^5,\ H_2^4), & (H_5^6,\ H_1^5,\ H_2^4), \\
& (H_5^6,\ H_2^5,\ H_2^4), & (H_5^6,\ H_3^5,\ H_1^4), & (H_5^6,\ H_3^5,\ H_2^4), \\
& (H_5^6,\ H_3^5,\ H_3^4), & (H_5^6,\ H_4^5,\ H_1^4), & (H_5^6,\ H_7^5,\ H_2^4), \\
& (H_6^6,\ H_1^5,\ H_1^4), & (H_6^6,\ H_1^5,\ H_2^4), & (H_6^6,\ H_2^5,\ H_1^4), \\
& (H_6^6,\ H_2^5,\ H_2^4), & (H_6^6,\ H_2^5,\ H_3^4), & (H_6^6,\ H_3^5,\ H_1^4), \\
& (H_6^6,\ H_3^5,\ H_2^4), & (H_6^6,\ H_3^5,\ H_3^4), & (H_6^6,\ H_4^5,\ H_2^4), \\
& (H_6^6,\ H_5^5,\ H_2^4), & (H_6^6,\ H_7^5,\ H_1^4), & (H_6^6,\ H_7^5,\ H_2^4), \\
& (H_6^6,\ H_7^5,\ H_3^4), & (H_7^6,\ H_1^5,\ H_2^4), & (H_7^6,\ H_2^5,\ H_2^4), \\
& (H_7^6,\ H_3^5,\ H_2^4), & (H_7^6,\ H_4^5,\ H_2^4), & (H_7^6,\ H_6^5,\ H_1^4), \\
& (H_7^6,\ H_6^5,\ H_2^4), & (H_7^6,\ H_7^5,\ H_2^4), & (H_7^6,\ H_7^5,\ H_3^4),
\end{array}
$$

$$
\begin{array}{lll}
(H_8^6,\ H_1^5,\ H_2^4), & (H_8^6,\ H_6^5,\ H_1^4), & (H_9^6,\ H_2^5,\ H_1^4), \\
(H_9^6,\ H_3^5,\ H_2^4), & (H_9^6,\ H_4^5,\ H_3^4), & (H_{10}^6,\ H_1^5,\ H_1^4), \\
(H_{10}^6,\ H_2^5,\ H_1^4), & (H_{10}^6,\ H_2^5,\ H_2^4), & (H_{10}^6,\ H_2^5,\ H_3^4), \\
(H_{10}^6,\ H_5^5,\ H_1^4), & (H_{10}^6,\ H_7^5,\ H_3^4), & (H_{11}^6,\ H_1^5,\ H_2^4), \\
(H_{11}^6,\ H_2^5,\ H_2^4), & (H_{11}^6,\ H_3^5,\ H_1^4), & (H_{11}^6,\ H_3^5,\ H_2^4), \\
(H_{11}^6,\ H_3^5,\ H_3^4), & (H_{11}^6,\ H_7^5,\ H_2^4)\}. &
\end{array}
$$

For $j_1 = 6,\ j_2 = 6,\ j_3 = 3$,

$$T=(G_1,G_2,G_3)\in \{(H^6_1,\ H^6_8,\ H^3_1),\quad (H^6_2,\ H^6_3,\ H^3_1),\quad (H^6_2,\ H^6_4,\ H^3_1),\\ (H^6_5,\ H^6_6,\ H^3_1),\quad (H^6_5,\ H^6_7,\ H^3_1),\quad (H^6_6,\ H^6_{11},\ H^3_1),\\ (H^6_7,\ H^6_{10},\ H^3_1)\}.$$

For $j_1=5,\ j_2=5,\ j_3=5$

$$T=(G_1,G_2,G_3)\in \{(H^5_1,\ H^5_2,\ H^5_3),\quad (H^5_1,\ H^5_2,\ H^5_6),\quad (H^5_1,\ H^5_2,\ H^5_7),\\ (H^5_1,\ H^5_3,\ H^5_5),\quad (H^5_1,\ H^5_3,\ H^5_7),\quad (H^5_1,\ H^5_5,\ H^5_7),\\ (H^5_2,\ H^5_3,\ H^5_4),\quad (H^5_2,\ H^5_3,\ H^5_5),\quad (H^5_2,\ H^5_3,\ H^5_7),\\ (H^5_2,\ H^5_5,\ H^5_6),\quad (H^5_2,\ H^5_6,\ H^5_7),\quad (H^5_3,\ H^5_4,\ H^5_7)\}.$$

C Multidesigns for K_7 and K_8

For the following, $V(K_7)=\mathbb{Z}_7$ and $V(K_8)=\mathbb{Z}_8$. We begin with T-multidecompositions of K_7 for $T=(H^7_i,H^5_j,H^3_1)$.

- $T=(H^7_4,H^5_2,H^3_1)$: $H^7_4\cong[3,4,5,0,1,2], H^5_2\cong[3,1,2,5,6,0]$,
  $H^3_1\cong[0,1,3,5,4,6],[0,2,3,6,4,5],[0,4,1,6,2,5]$
- $T=(H^7_5,H^5_2,H^3_1)$: $H^7_5\cong[3,4,5,0,1,2], H^5_2\cong[3,5,2,1,6,0]$,
  $H^3_1\cong[0,2,1,4,5,6],[0,4,2,5,3,6],[0,5,1,3,4,6]$
- $T=(H^7_5,H^5_3,H^3_1)$: $H^7_5\cong[3,4,5,0,1,2], H^5_3\cong[4,6,0,2,3,5]$,
  $H^3_1\cong[0,3,1,6,2,5],[0,4,1,3,2,6],[0,5,1,4,3,6]$
- $T=(H^7_5,H^5_7,H^3_1)$: $H^7_5\cong[3,4,5,0,1,2], H^5_7\cong[0,5,1,4,3,6]$,
  $H^3_1\cong[0,2,1,6,3,5],[0,3,2,5,4,6],[0,4,1,3,2,6]$
- $T=(H^7_6,H^5_2,H^3_1)$: $H^7_6\cong[3,1,5,0,4,2], H^5_2\cong[5,0,1,2,6,3]$,
  $H^3_1\cong[0,2,1,3,4,6],[0,3,1,4,5,6],[0,6,2,5,3,4]$
- $T=(H^7_8,H^5_3,H^3_1)$: $H^7_8\cong[2,1,5,0,4,3], H^5_3\cong[3,6,2,4,1,0]$,
  $H^3_1\cong[0,2,1,3,5,6],[0,3,2,5,4,6],[0,4,1,6,3,5]$

The T-multidecompositions of K_7 for $T=(H^6_i,H^6_j,H^3_1)$ are given by

- $T=(H^6_2,H^6_3,H^3_1)$: $H^6_2\cong[4,0,6,1,2,3]$,
  $H^6_3\cong[3,6,2,4,5,1]$, [5,3,4,6,0,2],
  $H^3_1\cong[0,5,1,4,2,6]$.
- $T=(H^6_2,H^6_4,H^3_1)$: $H^6_2\cong[3,4,6,0,1,2,]$, $H^6_4\cong[1,3,4,6,2,5]$,
  $H^3_1\cong[0,3,2,4,5,6]$, [0,5,1,4,3,6], [0,2,1,6,4,5].
- $T=(H^6_5,H^6_6,H^3_1)$: $H^6_5\cong[1,0,6,4,3,2]$, $H^6_6\cong[6,4,0,2,1,5]$,
  $H^3_1\cong[1,6,2,4,3,5]$, [1,3,2,6,0,5], [1,4,2,5,3,6].
- $T=(H^6_5,H^6_7,H^3_1)$: $H^6_5\cong[2,3,4,6,0,1]$, $H^6_7\cong[1,4,2,0,6,5]$,
  $H^3_1\cong[1,3,4,6,0,5]$, [1,4,3,6,2,5], [1,6,3,5,0,4].
- $T=(H^6_6,H^6_{11},H^3_1)$: $H^6_6\cong[1,2,3,4,6,0]$, $H^6_{11}\cong[4,5,3,0,1,6]$,
  $H^3_1\cong[1,4,2,6,0,5]$, [1,5,3,6,2,4], [1,6,2,5,0,4].
- $T=(H^6_7,H^6_{10},H^3_1)$: $H^6_7\cong[0,4,3,1,5,6]$, $H^6_{10}\cong[1,2,4,5,3,0]$,
  $H^3_1\cong[1,4,2,6,0,5]$, [1,5,2,4,3,6], [1,6,2,5,0,4].

We now move on to optimal T-multipackings of K_7 for $T=(H^8_i,H^4_j,H^3_1)$.

- $T = (H_1^8, H_1^4, H_1^3)$: $H_1^8 \cong [4,5,0,1,2,3]$,
  $H_1^4 \cong [1,5,2,3,6,4], [1,6,2,3,0,4]$,
  $H_1^3 \cong [0,6,1,3,2,4]$. Leave is 56 & 14.

- $T = (H_1^8, H_2^4, H_1^3)$: $H_1^8 \cong [4,5,0,1,2,3]$,
  $H_2^4 \cong [1,3,6,4,2,5], [0,6,1,5,2,4]$,
  $H_1^3 \cong [0,3,1,4,2,6]$. Leave is 04 & 56

- $T = (H_2^8, H_2^4, H_1^3)$: $H_2^8 \cong [4,5,0,1,2,3]$,
  $H_2^4 \cong [1,6,3,5,0,4], [4,6,2,5,1,3]$
  $H_1^3 \cong [0,2,1,4,5,6]$. Leave is 06 & 24.

- $T = (H_3^8, H_1^4, H_1^3)$: $H_3^8 \cong [4,0,1,3,6,2]$,
  $H_1^4 \cong [0,6,4,2,5,3], [0,5,1,2,3,4]$
  $H_1^3 \cong [0,2,1,6,4,5]$. Leave is 12 & 56.

- $T = (H_4^8, H_3^4, H_1^3)$: $H_4^8 \cong [4,5,0,1,2,3]$,
  $H_3^4 \cong [5,6,4,1,2,3], [3,1,4,0,2,6]$,
  $H_1^3 \cong [0,4,1,5,2,6]$. Leave is 06 & 35.

- $T = (H_5^8, H_1^4, H_1^3)$: $H_5^8 \cong [5,0,2,4,3,1]$,
  $H_1^4 \cong [0,6,4,2,5,3], [0,4,1,2,6,3]$,
  $H_1^3 \cong [1,6,2,3,4,5]$. Leave is 01 & 56.

The optimal T-multipackings of K_7 for $T = (H_i^7, H_j^5, H_1^3)$ are

- $T = (H_1^7, H_1^5, H_1^3)$: $H_1^7 \cong [4,5,0,1,2,3]$,
  $H_1^5 \cong [1,6,3,5,2,4], [5,3,1,2,0,6]$,
  $H_1^3 \cong [0,4,1,5,2,6]$. Leave is 46.

- $T = (H_2^7, H_1^5, H_1^3)$: $H_2^7 \cong [4,5,0,1,2,3]$,
  $H_1^5 \cong [5,6,3,2,4,1], [6,2,5,1,3,0]$,
  $H_1^3 \cong [0,4,1,6,3,5]$. Leave is 46.

- $T = (H_2^7, H_5^5, H_1^3)$: $H_2^7 \cong [4,5,0,1,2,3]$,
  $H_5^5 \cong [1,3,0,6,4,5], [2,5,0,4,1,6]$,
  $H_1^3 \cong [0,3,1,6,2,4]$. Leave is 36.

- $T = (H_3^7, H_1^5, H_1^3)$: $H_3^7 \cong [4,5,0,1,2,3]$,
  $H_1^5 \cong [6,5,2,4,1,3], [0,5,3,4,6,2]$,
  $H_1^3 \cong [0,3,1,6,2,4]$. Leave is 06.

- $T = (H_3^7, H_6^5, H_1^3)$: $H_3^7 \cong [4,5,0,1,2,3], H_6^5 \cong [0,2,3,5,4,6]$,
  $H_1^3 \cong [0,5,1,4,3,6], [0,3,1,6,2,5]$.
  Leave is 13, 26, & 56.

- $T = (H_9^7, H_4^5, H_1^3)$: $H_9^7 \cong [1,2,3,4,5,0]$,
  $H_4^5 \cong [0,1,6,4,5,3], [0,4,2,5,6,1]$,
  $H_1^3 \cong [0,6,1,5,3,4]$. Leave is 36.

- $T = (H_{10}^7, H_1^5, H_1^3)$: $H_{10}^7 \cong [1,2,3,4,5,0]$,
  $H_1^5 \cong [1,4,2,0,6,3], [2,3,0,1,6,5]$,
  $H_1^3 \cong [0,4,1,5,2,6]$. Leave is 46.

The optimal T-multicoverings of K_7 for $(T = H_i^7, H_j^5, H_1^3)$ are

- $T = (H_1^7, H_1^5, H_1^3)$: $H_1^7 \cong [4,5,0,1,2,3],[5,3,1,4,2,6]$,
  $H_1^5 \cong [4,6,3,5,2,0]$,
  $H_1^3 \cong [0,6,1,5,2,4]$. Padding is 24.
- $T = (H_2^7, H_1^5, H_1^3)$: $H_2^7 \cong [4,5,0,1,2,3],[5,3,0,4,6,1]$,
  $H_1^5 \cong [6,2,4,1,3,5]$,
  $H_1^3 \cong [1,4,2,5,3,6]$. Padding is 35.
- $T = (H_2^7, H_5^5, H_1^3)$: $H_2^7 \cong [4,5,0,1,2,3],[0,3,6,5,1,4]$,
  $H_5^5 \cong [2,4,1,3,5,6]$,
  $H_1^3 \cong [0,6,2,5,3,4]$. Padding is 34.
- $T = (H_3^7, H_1^5, H_1^3)$: $H_3^7 \cong [4,5,0,1,2,3],[3,1,5,6,2,0]$,
  $H_1^5 \cong [4,6,3,0,5,2]$,
  $H_1^3 \cong [0,6,1,4,2,3]$. Padding is 23.
- $T = (H_3^7, H_6^5, H_1^3)$: $H_3^7 \cong [3,2,0,1,5,4],[0,2,5,6,1,4]$,
  $H_6^5 \cong [3,6,2,5,4,1]$,
  $H_1^3 \cong [0,6,4,2,3,5]$. Padding is 14.
- $T = (H_9^7, H_4^5, H_1^3)$: $H_9^7 \cong [1,2,3,4,5,0],[0,1,4,5,2,6]$,
  $H_4^5 \cong [0,6,3,1,4,5]$,
  $H_1^3 \cong [0,4,1,5,2,3]$. Padding is 23.
- $T = (H_{10}^7, H_1^5, H_1^3)$: $H_{10}^7 \cong [1,2,3,4,5,0]$,
  $H_1^5 \cong [4,1,5,3,6,2],[4,6,5,1,3,0]$,
  $H_1^3 \cong [0,6,1,4,2,3],[1,6,2,5,3,4]$.
  Padding is 14 & 34.

Finally, we have the following optimal T-multipackings of K_8 for $T = (H_i^6, H_j^6, H_1^3)$.

- $T = (H_1^6, H_8^6, H_1^3)$: $H_1^3 \cong [1,6,2,4,5,7],[0,2,1,3,4,7],[1,7,3,5,4,6]$,
  $[1,4,3,6,2,7],[0,4,1,5,6,7]$.
  $H_1^6 \cong [0,1,2,3,4,5]$, $H_8^6 \cong [0,3,2,5,6,7]$.
  Leave is 06.
- $T = (H_2^6, H_3^6, H_1^3)$: $H_1^3 \cong [1,6,2,4,5,7],[0,2,3,7,4,6],[1,7,2,6,3,5]$,
  $[2,7,3,6,4,5],[0,3,2,5,4,7]$
  $H_2^6 \cong [3,4,6,0,1,2]$, $H_3^6 \cong [7,0,4,1,5,6]$.
  Leave is 13.
- $T = (H_2^6, H_4^6, H_1^3)$: $H_1^3 \cong [0,3,1,5,4,7],[0,7,2,6,3,5],[1,7,3,6,4,5]$,
  $[0,5,2,7,4,6],[0,2,1,6,3,7]$,
  $H_2^6 \cong [2,1,6,0,4,3]$, $H_4^6 \cong [6,7,1,4,2,5]$.
  Leave is 13.
- $T = (H_5^6, H_6^6, H_1^3)$: $H_1^3 \cong [0,6,1,5,4,7],[0,2,3,7,4,6],[0,4,1,3,2,6]$,
  $[0,5,1,4,2,7],[1,7,2,5,3,6]$,
  $H_5^6 \cong [2,3,4,7,0,1]$, $H_6^6 \cong [7,5,4,2,1,6]$.
  Leave is 35.
- $T = (H_5^6, H_7^6, H_1^3)$: $H_1^3 \cong [0,2,1,6,3,7],[0,4,1,5,2,7],[0,5,1,3,2,4]$,
  $[1,7,2,6,3,4],[1,4,3,6,5,7]$,
  $H_5^6 \cong [1,0,6,5,3,2]$, $H_7^6 \cong [0,4,5,2,6,7]$.
  Leave is 46.

- $T = (H_6^6, H_{11}^6, H_1^3)$: $H_1^3 \cong [0,3,2,7,4,6], [0,5,1,3,4,7], [0,6,1,5,2,4],$
  $[1,7,2,6,4,5], [1,4,2,5,3,6],$
  $H_6^6 \cong [5,7,0,4,1,6], H_{11}^6 \cong [1,2,3,4,5,0]$.
  Leave is 37.
- $T = (H_7^6, H_{10}^6, H_1^3)$: $H_1^3 \cong [0,4,3,6,5,7], [0,5,1,3,2,7], [0,6,2,4,3,7],$
  $[0,7,1,6,2,5], [1,5,2,6,3,4],$
  $H_7^6 \cong [1,4,5,3,6,7], H_{10}^6 \cong [1,0,4,6,3,2]$.
  Leave is 14.