A DISCRETE HOMOTOPY THEORY FOR GRAPHS, WITH APPLICATIONS TO ORDER COMPLEXES OF LATTICES

by

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ABSTRACT

A-theory is a recently developed area of algebraic combinatorics that takes concepts from algebraic topology and transfers them to a combinatorial setting. This dissertation focusses on a discrete homotopy theory defined on simple graphs and simplicial complexes. Early appearances of this concept of discrete homotopy can be found in the work of Atkin, and, later on, Malle. More recently, Laubenbacher and Kramer became aware of Atkin's work while conducting research in social and communications networks. With Barcelo and Weaver, they pursued Atkin's ideas and extended them to form what is now known as A-theory. This theory provides a general framework encompassing homotopy methods that can be used to prove connectivity results for graphs and matroids, for example.

In this dissertation, it is shown that the discrete fundamental group of the box product of two graphs is isomorphic to the direct product of the discrete fundamental groups of the individual graphs. Isomorphisms of the discrete fundamental group of a graph, Γ, to the free product of the discrete fundamental groups of a collection of subgraphs of Γ are presented in the cases where Γ has a single cut vertex or a minimal cut set of size two. A graph may also arise in relation to the order complex of a lattice. Results are proven for the discrete fundamental group of a graph related to the order complex of the direct sum, ordinal sum, or ordinal product of two finite graded lattices. A construction is given of the graph related to the order complex of the direct product of two finite graded lattices. This is explored further in the case of the Boolean lattice, which may be viewed as the direct product of smaller lattices. The abelianization of the discrete fundamental group of the order complex of the Boolean lattice is a free group on $2^{n-3}(n^2 - 5n + 8) - 1$ generators, which recovers a formula from Björner and Welker in their work on the computational complexity of the k-equal problem, a computer science application.

This dissertation is dedicated to my grandmothers,

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TABLE OF CONTENTS

Page

LIST OF FIGURES

Figure Page

CHAPTER 1

INTRODUCTION

A-theory is a recently developed area of algebraic combinatorics that takes concepts from algebraic topology and transfers them to a combinatorial setting. A-theory contains discrete analogues to continuity, homotopy, and fundamental group, defined on graphs and simplicial complexes. This dissertation focusses on a discrete homotopy theory defined on simple graphs. An early appearance of this concept of discrete homotopy can be found in the work of Atkin [1, 2] in the early 1970s. A physicist modeling social networks using simplicial complexes, Atkin developed Q-analysis, a discrete topological theory used to measure the combinatorial connectivity of a complex and identify combinatorial "holes" in the complexes. In 1972, Maurer [13] developed a similar concept of discrete deformation of paths in graphs while working on his dissertation, developing a characterization of basis graphs of matroids. In 1983, Malle [12] also defined a notion of equivalence of graph maps, as well as discrete fundamental group. He was able to characterize graphs with a trivial discrete fundamental group, and also showed that this group corresponds to the classical fundamental group when the girth of a graph, the length of the shortest cycle in the graph, is at least five. These authors were apparently unaware of each other's work, but in fact the concepts they created are all equivalent. More recently, Laubenbacher and Kramer [11] became aware of Atkin's work while conducting research in social and communications networks. With Barcelo and Weaver [3], they pursued Atkin's ideas in Q-analysis and extended them to include graphs and discrete analogues to higher homotopy groups. They also named their work A-theory in his honor.

In Chapter 2, we review Barcelo, Kramer, Laubenbacher, and Weaver's [3] definitions of a discrete homotopy theory for graphs. For simple graphs Γ and Γ', a *graph map* $f : \Gamma \to \Gamma'$ maps vertices to vertices and preserves adjacency, that is, the pair of vertices incident to an edge in Γ are mapped to a single vertex, or to a pair of adjacent vertices in Γ'. In classical topology, two continuous maps from one topological space to another are homotopic to one another if one can be continuously deformed into the other. For graph maps that send vertices to vertices, they define two maps to be equivalent if one can be discretely deformed into the other, while preserving adjacency. This relation, called *G-homotopy*, is an equivalence relation on the set of graph maps from Γ to Γ'.

In particular, Barcelo et al. consider graph maps defined on I, a discrete analogue to $[0, 1]$. Given a distinguished vertex $v_0 \in \Gamma$, the image of a *based graph map* $f : I \to \Gamma$ with only finitely many values not equal to v_0 is a *loop* in Γ based at v_0. The set of equivalence classes of graph maps based at v_0, with multiplication of equivalence classes corresponding to the concatenation of loops in Γ, is their *discrete fundamental group* $A_1^G(\Gamma, v_0)$, which we also refer to as the G-group of Γ. In classical topology, the fundamental group of a cycle is isomorphic to $\mathbb{Z}$. We give another proof of a theorem from Barcelo et al. [3], demonstrating that the discrete fundamental group of a cycle of length three or four is trivial, and is isomorphic to $\mathbb{Z}$ if the cycle is of length at least five.

In Chapter 3, we look for ways to make computing the discrete fundamental group of a graph easier by exploring methods for decomposing a graph Γ into subgraphs, computing the discrete fundamental groups of the smaller graphs, then using these groups to find the discrete fundamental group of Γ. We prove a discrete analogue of a theorem from classical topology: the discrete fundamental group of $\Gamma \Box \Gamma'$, the box product of Γ and Γ', is isomorphic to the direct product of the G-groups of Γ and Γ':

$$A_1^G(\Gamma \Box \Gamma', (v_0, v_0')) \simeq A_1^G(\Gamma, v_0) \times A_1^G(\Gamma', v_0').$$

We also use a Seifert-Van Kampen type theorem from Barcelo et al. [3] to prove results in the case where Γ has a cut vertex or a minimal cut set of size two.

Barcelo et al. [3] have developed a discrete homotopy theory for simplicial complexes as well. In this dissertation, we consider one type of simplicial complex, the order complex of a finite graded lattice. To compute the discrete fundamental group of the order complex of a graded lattice L of rank k, denoted by $A_1^{k-3}(\Delta(\overline{L}))$ or simply referred to as the A_1 group of the lattice, we first construct a graph $\Gamma_{max}^{k-3}(\Delta(\overline{L}))$. The vertices of the graph correspond to maximal chains in $\overline{L} = L - \{\hat{0}, \hat{1}\}$, the truncated lattice, and two vertices are adjacent if the corresponding chains differ in precisely one element. Barcelo et al. [3] showed that the G-group of the graph is isomorphic to the discrete fundamental group of the order complex. We may then build new lattices from smaller ones, and we look for relationships between the discrete fundamental groups of the lattices by considering the structure of the related graphs. In Chapter 4, we prove results for the direct sum, ordinal sum, and ordinal product of two graded lattices. We also provide a construction for the graph associated to the direct product of two graded lattices.

While there does not appear to be a simple relationship between the A_1 groups of lattices L_1, L_2, and the direct product $L_1 \times L_2$; in Chapter 5 we use the structure of the graph introduced in Chapter 4 and the fact that B_n, the Boolean lattice of rank n, may be expressed as the product of smaller lattices, to obtain the main result of this dissertation. This result, computing the number of generators of $A_1^{n-3}(\Delta(\overline{B_n}))^{ab}$, the abelianization of the discrete fundamental group of the Boolean lattice, is related to the k-equal problem, a computational complexity problem studied by Björner, Lovász, and Welker [6, 7]: Given n real numbers and an integer $k \geq 2$, how many comparisons are needed to determine if k of the numbers are equal? We can reframe this as an equivalent geometric problem. Let $V_{n,k}$ be the set of points in $\mathbb{R}^n$ that have at least k equal coordinates. Given a point in $\mathbb{R}^n$, what is the complexity of determining if that point is in $V_{n,k}$? Björner and Lovász [6] showed that the Betti numbers of $M_{n,k}$, the complement of $V_{n,k}$ in $\mathbb{R}^n$, are essential for determining a lower bound for the complexity of our geometric problem when using a linear decision tree model.

Björner and Welker [7] computed these Betti numbers using algebraic and topological methods. Using A-theory and purely combinatorial methods, we compute the first Betti number of $M_{n,3}$, using a relationship between $M_{n,3}$ and B_n. It is known that the graph $\Gamma^{n-3}_{max}(\Delta(\overline{B_n}))$ is the 1-skeleton of the permutahedron P_{n-1} [17]. (For ease of notation, we refer to this graph as Γ_{B_n}.) Barcelo et al. [3] showed that if 2-cells are attached to the 3- and 4-cycles of a graph Γ, the classical fundamental group of the resulting cell complex, is isomorphic to the discrete fundamental group of Γ. Babson [4] observed that attaching 2-cells to the 4-cycles of Γ_{B_n} yields a cell complex that is homotopy equivalent to $M_{n,3}$. Furthermore, thanks to Björner and Welker [7], we know that the first homology group of $M_{n,3}$ is a free group, thus to find the first Betti numbers we only need to compute the rank of $A_1^{n-3}(\Delta(\overline{B_n}))^{ab}$. We see that if we attach 2-cells to the 4-cycles in Γ_{B_n}, we are left with 6-cycles, so our goal is to find a way to define equivalence classes of graph maps whose images are 6-cycles in the graph.

The Boolean lattice B_n is isomorphic to $B_{n-1} \times \mathbf{2}$, where $\mathbf{2}$ is the poset on two elements, x and y, with $x < y$. This isomorphism, combined with the construction of the graph associated to a direct product of lattices defined in Chapter 4, give us a better understanding of the structure of Γ_{B_n}. The vertices of Γ_{B_n} correspond to permutations in S_n, and two vertices are adjacent if the permutations differ by right multiplication by a simple transposition. That is, $v \sim v' \iff \sigma_v = \sigma_{v'}(i, i+1)$ for some i, $1 \leq i \leq n-1$. Thus Γ_{B_n} is bipartite, with the set of vertices partitioned into even and odd permutations, and each edge in the graph corresponds to a simple transposition in S_n. The edges in a 4-cycle in Γ_{B_n} correspond to a pair of disjoint simple transpositions in S_n. A *reduced* 6-cycle cannot be expressed as the concatenation of two 4-cycles, and its edges correspond to a pair of transpositions of the form $(i-1, i)$ and $(i, i+1)$ for some i, $1 \leq i \leq n-1$.

In Theorem 6.2, the main result of this dissertation, we prove that two reduced 6-cycles in Γ_{B_n}, C_1 and C_2, are G-homotopic to one another if and only if they correspond to the same pair of transpositions and they differ by a sequence of transpositions, that is $C_2 = C_1\tau_1 \ldots \tau_k$ where the τ_j are simple transpositions in S_n that are disjoint from $(i-1, i)$ and $(i, i+1)$. This theorem gives us the means to describe and enumerate the equivalence classes of reduced 6-cycles in Γ_{B_n}, yielding the result that the rank of $A_1^{n-3}(\Delta(\overline{B_n}))^{ab}$, and thus also the Betti number we wanted to compute, is $2^{n-3}(n^2 - 5n + 8) - 1$, recovering a formula from Björner and Welker's work [7] on the k-equal problem.

CHAPTER 2

G-HOMOTOPY OF GRAPHS

2.1. G-homotopy of Graph Maps

In this section we review the basic concepts of a discrete homotopy theory for graphs, originally developed by Laubenbacher and Kramer and then later extended with Barcelo and Weaver. These concepts are discrete analogues of continuity, homotopy, and fundamental group, and can be found in greater detail in [11]. For the sake of completeness, we present many of the details here. Let $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ be *simple* graphs, with no loops or parallel edges. A *graph map* $f : \Gamma \to \Gamma'$ is a set map $V \to V'$ that preserves adjacency, that is, if $vw \in E$, then either $f(v)$ is adjacent to $f(w)$ in Γ', denoted by $f(v) \sim_{\Gamma'} f(w)$, or $f(v) = f(w)$. Let $v \in V$ and $v' \in V'$ be distinguished vertices. A *based graph map* is a graph map $f : (\Gamma, v) \to (\Gamma', v')$ such that $f(v) = v'$. The *box product* $\Gamma \Box \Gamma'$ of two graphs, Γ and Γ', is the graph with vertex set $V \times V'$ and an edge between (v, v') and (w, w') if either

1. $v = w$ and $v' \sim_{\Gamma'} w'$, or

2. $v' = w'$ and $v \sim_{\Gamma} w$.

We note that if Γ is connected, then the image of f is a connected subgraph of Γ'.

Definition 2.1. (Kramer and Laubenbacher [11])

1. *Let* $\Gamma = (V, E)$ *and* $\Gamma' = (V', E')$ *be simple graphs with distinguished vertices* $v_0, v_1 \in V$ *and* $v_0', v_1' \in V'$. *Let* f, g *be based graph maps* $\Gamma \to \Gamma'$ *such that* $f(v_0) = g(v_0) = v_0'$ *and* $f(v_1) = g(v_1) = v_1'$. *We say that* f *and* g *are G-homotopic relative to* v_0' *and* v_1', *denoted by* $f \simeq_G g \quad rel(v_0', v_1')$, *if there is an integer* n *and a graph map* $F : \Gamma \Box I_n \to \Gamma'$ *such that*

   (a) $F(v, 0) = f(v) \qquad \forall\, v \in V$

   (b) $F(v, n) = g(v) \qquad \forall\, v \in V$

   (c) $F(v_0, j) = v_0' \qquad 0 \leq j \leq n$

   (d) $F(v_1, j) = v_1' \qquad 0 \leq j \leq n.$

   If $v_0' = v_1'$, *then we write* $f \simeq_G g \quad rel(v_0')$, *or simply* $f \simeq_G g$ *of the base vertex is clear.*

2. *We call* (Γ, v_0) *and* (Γ', v_0') *G-homotopy equivalent if there exist based graph maps* $f : \Gamma \to \Gamma'$ *and* $g : \Gamma' \to \Gamma$ *such that* $g \circ f \simeq_G id_\Gamma \quad rel(v_0)$ *and* $f \circ g \simeq_G id_{\Gamma'} \quad rel(v_0')$. *The maps* f *and* g *are called G-homotopy inverses of each other.*

3. *If* Γ' *is a subgraph of* Γ *with base vertex* $v_0 \in \Gamma'$, *then* Γ' *is called a G-homotopy retract of* Γ *if there exists a based G-homotopy inverse of the inclusion map. This G-homotopy inverse is called a G-homotopy retraction.*

Let I_m be the path on $m + 1$ vertices labelled $0, 1, 2, ..., m$ with edges $(i - 1)i$ for $1 \leq i \leq m$. This "discrete interval" plays a similar role to that of the unit interval in classical homotopy theory. Let $f : I_m \to \Gamma$ be a graph map such that $f(0) = v_0$ and

$f(m) = v_1$. Then the image of I_m under f is a *string*, a sequence of vertices, beginning with v_0 and ending with v_1, in which a pair of consecutive vertices are adjacent or identical in Γ. If $v_0 = v_1$, then the image is a *string loop*, or simply a *loop*, based at v_0. For the sake of simplicity, when referring to such a graph map, we often simply refer to its image in Γ. In this dissertation, the domain of our graph maps is often the graph I_m, and we work specifically with graph maps $I_m \to \Gamma$ whose images have the same endpoints in Γ. The graph map F defined above is called a G-homotopy from f to g, relative to (v_0, v_1); it is the discrete deformation of f into g. When f and g are defined on I_m, one can also visualize the graph $I_m \square I_n$ as an $(m+1) \times (n+1)$ grid, with vertex (i, j) in the grid labelled by $F(i, j)$. F is a graph map, so if there is an edge between two vertices in the grid then the images of the vertices must be adjacent or identical in Γ. Furthermore, the image of row j of the grid corresponds to $\{F(i, j) : i \in I_m\}$, and is a string F_j from v_0 to v_1 in Γ.

A graph map $f : I_m \to \Gamma$ can be extended to a graph map $f' : I_p \to \Gamma$ for all $p > m$ by sending all vertices $j > m$ to $f(m)$. Thus two graph maps defined on I_m and I_p, respectively, can be viewed as being defined on the larger discrete interval. The ability to "stretch" a graph map enables us to discuss the set of graph maps whose images are strings in Γ of finite length, rather than limiting us to maps defined on a discrete interval of a fixed length. Let I be the infinite path with vertices corresponding to the set of non-negative integers and edges corresponding to a pair of consecutive integers. We can define a relation on the set of graph maps f from I to Γ, where each map f satisfies the conditions that $f(0) = v_0$ and there is a positive integer m_f such that $f(j) = v_1$ for $j \geq m_f$. The image of each of these graph maps will be a string in Γ of finite length, beginning at v_0 and ending at v_1. We note that if one restricts f to the discrete interval I_{m_f}, the image will be the same string from v_0 to v_1 in Γ. In proofs involving graph maps from I to Γ, the notation is often easier if we consider the restriction of the maps to a path of finite length, thus, when we are discussing a graph map $f : I \to \Gamma$, we will write $f : I_m \to \Gamma$, indicating the restriction of f to I_{m_f}. Furthermore, if we have a G-homotopy between two graph maps f and g and we describe the G-homotopy grid, we write $f, g : I_m \to \Gamma$, and we assume that any necessary stretching has been done so that both (restricted) maps are defined on intervals of the same length.

Figure 1. A G-homotopy from f to g.

Examples. (1) ***G*-homotopy.** Figure 1 gives an example of a G-homotopy between two based graph maps defined on intervals of different lengths. The graph map g has been stretched so that it is defined on the longer interval.

Figure 2. The retraction of a 4-cycle to a single vertex.

(2) ***G*-homotopy retract.** [11] Let Γ be a 4-cycle with vertices $\{v_0, v_1, v_2, v_3\}$, and let Γ' be the subgraph of Γ consisting of the single vertex v_0. Figure 2 illustrates a graph map $f : \Gamma \square I_2 \to \Gamma$ that will contract the 4-cycle to a single vertex. On $\Gamma \square \{0\}$, f is the identity. On $\Gamma \square \{1\}$, define f by $f(v_0, 1) = f(v_3, 1) = v_0$ and $f(v_1, 1) = f(v_2, 1) = v_1$. On $\Gamma \square \{2\}$, all vertices are mapped to v_0. It requires a minimum of two steps to perform this contraction, because if one attempted to contract Γ to v_0 with a map $g : \Gamma \square I_1 \to \Gamma$, this would require mapping $(v_2, 0)$ to v_2 and $(v_2, 1)$ to v_0. However, $(v_2, 0)$ and $(v_2, 1)$ are adjacent in $\Gamma \square I_1$, but v_2 and v_0 are not adjacent in Γ, so g would not be a valid graph map.

A 3-cycle can be similarly contracted to a single vertex, although that may be done in a single step.

(3) **Gangster Problem.** Malle [12] gives an interesting justification of why it is not possible to perform a similar contraction of a 5-cycle. In his Gangster Problem, he describes an interpretation of the vertices of a graph as towns, and the edges as roads between the towns. Suppose there is a gangster living in each town, and they would like to have a meeting in a single town. For safety reasons they decide:

1. Each day, each gangster will move to an adjacent town or rest in the same town.
2. If two gangsters are in adjacent towns initially, they must be in adjacent towns or the same town each day.

Figure 3. The Gangster Problem

For what graphs is it possible for them to meet? We can see that this is equivalent to being able to discretely contract a graph to a single vertex. Moving (at most) once each day corresponds to increments in our discrete interval I_m. The second precaution corresponds to preserving adjacency within a G-homotopy. In Figure 3, the first graph is a 5-cycle with an additional edge, and it can be contracted to a single vertex. However, for the second graph, a 5-cycle, if one tries to construct a G-homotopy $f : \Gamma \Box I_m \to \Gamma$ as in Example 2 previously, one finds that in order to preserve adjacency at each level, f must map $\Gamma \Box (i)$ *onto* Γ for each i, $1 \leq i \leq m$, and it is not possible to discretely contract the image to a single vertex. The same is true for longer cycles as well, and thus any cycle of length ≥ 5 cannot be contracted to a single vertex.

Proposition 2.2. [11] *G-homotopy relative to (v_0, v_1) is an equivalence relation on the set of based graph maps $f : I \to \Gamma$ satisfying the conditions that*

1. *$f(0) = v_0$, and*

2. *there is a positive integer m_f such that $f(i) = v_1$ for $i \geq m_f$.*

Proof. To prove this, one views a G-homotopy F between graph maps as the $I_m \square I_n$ grid with vertex (i, j) labelled by $F(i, j)$.

1. Clearly a graph map $f : I_m \to \Gamma$ is G-homotopic to itself $rel(v_0, v_1)$. A G-homotopy grid would only require a single row.

2. Suppose $f \simeq_G g \quad rel(v_0, v_1)$. Then there exists a G-homotopy grid from f to g. Flipping this grid upside down yields a G-homotopy grid from g to f which shows $g \simeq_G f \quad rel(v_0, v_1)$.

3. Let h be yet another graph map $I_m \to \Gamma$, such that $g \simeq_G h \quad rel(v_0, v_1)$, and consider G-homotopy grids for $f \simeq_G g \quad rel(v_0, v_1)$ and $g \simeq_G h \quad rel(v_0, v_1)$. The last row of the grid for $f \simeq_G g \quad rel(v_0, v_1)$ and the first row of the grid for $g \simeq_G h \quad rel(v_0, v_1)$ both correspond to the the image of graph map g. Since all three graph maps are defined on the interval I_m, extending one or more of the maps if necessary, both grids have the same width. Connect the grids by "pasting" them along the row corresponding to g, and the result is a G-homotopy grid that shows $f \simeq_G h \quad rel(v_0, v_1)$.

$\square$

We now have G-homotopy classes $[f]$ of graph maps $f : I \to \Gamma$ whose images are strings of finite length from v_0 to v_1, under the equivalence relation $\simeq_G \ rel(v_0, v_1)$. Next we recall the operation on these classes. If $f : I_m \to \Gamma$ is a graph map with $f(0) = v_0$ and $f(m) = v_1$, and $g : I_k \to \Gamma$ is a graph map with $g(0) = v_1$ and $g(k) = v_2$, the product of these graph maps is a graph map $fg : I_{m+k} \to \Gamma$ given by

$$fg(i) = \begin{cases} f(i) & 0 \leq i \leq m \\ g(i-m) & m \leq i \leq m+k. \end{cases}$$

The image of I_{m+k} under fg is a string in Γ from v_0 to v_2 going through v_1. So it is first "string f" and then "string g". This operation is well-defined on equivalence classes:

Proposition 2.3. [11] *Let f, g be graph maps as defined above. If $f \simeq_G f' \quad rel(v_0, v_1)$ and $g \simeq_G g' \quad rel(v_1, v_2)$, then $fg \simeq_G f'g' \quad rel(v_0, v_2)$.*

Proof. Suppose $F : I_m \square I_n \to \Gamma$ is a G-homotopy from f to f', and $G : I_k \square I_l \to \Gamma$ is a G-homotopy from g to g'. Without loss of generality, suppose $l \leq n$, then define a new G-homotopy $G' : I_k \square I_n \to \Gamma$ by

$$G'(i,j) = \begin{cases} G(i,j) & 0 \leq j \leq l \\ G(i,l) & l \leq j \leq n \end{cases}$$

This has the effect of increasing the length of the G-homotopy grid for G by adding copies of the last row so that it will have the same number of rows as the grid for F. The two grids can be pasted together along the last column of the grid for F and the first column of the grid for G'. The resulting $(m+k+1) \times (n+1)$ grid corresponds to a new graph map which is denoted by $FG' : I_{m+k} \square I_n \to \Gamma$ and defined by

$$FG'(i,j) = \begin{cases} F(i,j) & 0 \leq i \leq m \\ G'(i-m,j) & m \leq i \leq m+k. \end{cases}$$

It is easy to verify that FG' is a G-homotopy from fg to $f'g' \quad rel(v_0, v_1)$. Thus multiplication of the equivalence class of f on the right by the equivalence class of g is well-defined: $[f][g] = [fg]$. $\square$

Of particular interest to us is the collection of graph maps whose images are loops in a graph Γ based at a fixed vertex, v_0. The boundary ∂I_m of I_m is the vertex set $\{0, m\}$. Let $f : (I_m, \partial I_m) \to (\Gamma, v_0)$ be a based graph map such that $f(\partial I_m) = v_0$. Then the image of I_m under f is a loop in Γ. The image of the product of two such graph maps is also a loop in Γ based at v_0, so we can now view multiplication as a group operation on the set of equivalence classes of graph maps whose images are loops based at v_0.

Theorem 2.4. ([11]) *Let $A_1^G(\Gamma, v_0)$ be the set of G-homotopy classes of based graph maps $\{f : I \to \Gamma\}$ satisfying the conditions that*

1. *$f(0) = v_0$, and*

2. *there is a positive integer m_f such that $f(i) = v_0$ for $i \geq m_f$.*

If multiplication in $A_1^G(\Gamma, v_0)$ is defined as above, then $A_1^G(\Gamma, v_0)$ becomes a group.

Proof. The identity of $A_1^G(\Gamma, v_0)$ is the equivalence class of the constant map that sends all vertices in I_m to the base vertex v_0. It is easy to see that for graph maps f, g, and $h : (I_m, \partial I_m) \to (\Gamma, v_0)$, $(fg)h$ and $f(gh)$ will result in the same concatenation of loops in Γ, and so multiplication of the corresponding equivalence classes is associative. The inverse of a class $[f]$ is the class of f^{-1} defined by $f^{-1}(i) = f(m - i)$, $0 \leq i \leq m$. The image of f^{-1} is the loop of f traversed in the opposite direction. To demonstrate that $ff^{-1} \simeq_G e_{(\Gamma, v_0)}$, define $F : I_{2m} \Box I_m \to (\Gamma, v_0)$ by

$$F(i,j) = \begin{cases} f(i) & 0 \leq i \leq m-j \\ f^{-1}(j) = f(m-j) & m-j \leq i \leq m+j \\ f^{-1}(i-m) = f(2m-i) & m+j \leq i \leq 2m. \end{cases}$$

Figure 4 gives an example of the G-homotopy defined above. Similarly, the graph map G defined by

$$G(i,j) = \begin{cases} f^{-1}(i) & 0 \leq i \leq m-j \\ f(j) = f^{-1}(m-j) & m-j \leq i \leq m+j \\ f(i-m) = f^{-1}(2m-i) & m+j \leq i \leq 2m. \end{cases}$$

is a G-homotopy from $f^{-1}f$ to the constant map $e_{(\Gamma,v_0)}$. □

Figure 4. A G-homotopy from ff^{-1} to the identity map $e_{(\Gamma,v_0)}$.

Is there a relation between $A_1^G(\Gamma, v_0)$ and $A_1^G(\Gamma, v_1)$? Not if v_0 and v_1 lie in different components of Γ. However, the following proposition shows that if Γ is connected, the group $A_1^G(\Gamma, v_0)$ is independent of the choice of v_0, up to isomorphism. In that case, we often write simply $A_1^G(\Gamma)$ for $A_1^G(\Gamma, v_0)$, and call it the *discrete fundamental G-group* of Γ.

Proposition 2.5. [11] *Let v_0, v_1 be two vertices lying in the same component of Γ. Let $[f]$ be an equivalence class of based graph maps in $A_1^G(\Gamma, v_0)$. Let $g : I_m \to \Gamma$ be a graph map such that $g(0) = v_0$ and $g(m) = v_1$. (The image of g is a string from v_0 to v_1 in Γ.) The mapping $[f] \to [g^{-1}fg]$ is an isomorphism g_* of the group $A_1^G(\Gamma, v_0)$ onto $A_1^G(\Gamma, v_1)$.*

Proof. Let $[f]$ and $[h]$ be two equivalence classes in $A_1^G(\Gamma, v_0)$, and let g be defined as above. It is easy to verify that

$$\begin{aligned} g_*[fh] &= [g^{-1}fhg] \\ &= [g^{-1}fgg^{-1}hg] \\ &= [g^{-1}fg][g^{-1}hg] \\ &= g_*[f]g_*[h] \end{aligned}$$

so g_* is a homomorphism and $(g^{-1})_*$ is its inverse. $\square$

2.2. The Discrete Fundamental Group of a Cycle

Figure 5. A cycle C_k

Inspired by arguments from classical homotopy theory (for example, see Greenberg [10]), we give another proof to show that the discrete fundamental group of a cycle of length ≥ 5 is isomorphic to $\mathbb{Z}$. Let C_k be the graph on k vertices, $k \geq 3$, labelled $0, 1, 2, ..., k-1$ counterclockwise as shown in Figure 5, and with edges $(k-1)0$ and $(i-1)i$ for $1 \leq i \leq k-1$. As we saw in the previous section, $A_1^G(C_k, 0)$ is trivial for $k = 3$ and $k = 4$. For $k \geq 5$, it is well known (see, for example, Malle [12], and Barcelo et al. [3]) that the homotopy class of a based graph map $f : (I_m, \partial I_m) \to C_k$ can be determined by the number of times the image of f "winds around" the cycle, with the number being negative if the "winding" is clockwise on C_k. Thus two graph maps are G-homotopic to each other if their images ultimately wind around C_k the same number of times in the same direction. To see this, we first define a map $\Phi : \mathbb{Z} \to A_1^G(C_k, 0)$ and then prove that it is an isomorphism.

Step 1. Definition of Φ.

Let $\Phi : \mathbb{Z} \to A_1^G(C_k, 0)$, where $k \geq 5$, be given by $\Phi(w) = [f_w]$, where f_w is the graph map $f_w : I_{|wk|} \to C_k$, $f_w(i) = sgn(w)i \pmod{k}$. For $w \geq 0$ the image of f_w winds around C_k w times in the counterclockwise direction. If $w < 0$, the winding is in the clockwise direction.

Step 2. Show Φ is a homomorphism.

Let $w, x \in \mathbb{Z}$. The image of f_{w+x} is a loop that winds around C_k $w + x$ times. Clearly, this is equivalent to the concatenation of a loop that winds around w times and a loop that winds around x times. Thus

$$\begin{aligned}\Phi(w+x) &= [f_{w+x}] \\ &= [f_w][f_x] \\ &= \Phi(w)\Phi(x).\end{aligned}$$

Step 3. Show Φ is surjective.

Let $f : (I_m, \partial I_m) \to (C_k, 0)$ be a based graph map. We will show that there is exists an integer w_f such that $\Phi(w_f) = [f]$. To do this we will define a new function $\rho : \{0, 1, 2, ...m\} \to \mathbb{Z}$ recursively as follows:

$$\rho(i) = \begin{cases} 0 & \text{if } i = 0 \\ \rho(i-1)+1 & \text{if } f(i) \equiv f(i-1)+1 \pmod{k} \\ \rho(i-1)-1 & \text{if } f(i) \equiv f(i-1)-1 \pmod{k} \\ \rho(i-1) & \text{if } f(i) = f(i-1). \end{cases}$$

For each i, $0 \leq i \leq m$, we have $\rho(i) \equiv f(i) \pmod{k}$, and $\rho(i)$ increases as the image of f winds counterclockwise around C_k, and decreases as f winds in the clockwise direction. If we divide $\rho(m)$ by k, the length of the cycle, the result will be an integer yielding the number of times (with direction) that f winds around C_k. Therefore we will define w_f to be $\frac{\rho(m)}{k}$. Recall that a graph map that winds once around C_k clockwise, then once counterclockwise, is G-homotopic the the trivial map, so one winding around C_k in one direction will "cancel out" one winding in the opposite direction when we count the total number of times the image of a graph map winds around the cycle. To show that w_f is well-defined, let $g : (I_m, \partial I_m) \to C_k$ be another representative of $[f]$ and let $F : I_m \Box I_n \to C_k$ be a G-homotopy from f to g. Let $\rho_j(i) = \rho(F(i,j))$ be the function ρ evaluated on the image of row j of the grid. Consider the first two rows of the G-homotopy grid for F. Recall that from the definition of ρ we have $\rho_0(0) = \rho_1(0) = 0$, as well as $\rho_0(i) \equiv F(i,0) \pmod{k}$ and $\rho_1(i) \equiv F(i,1) \pmod{k}$. Furthermore, for each i, $0 \leq i \leq m$, $F(i,0)$ and $F(i,1)$ differ by at most $1 \pmod{k}$, and consequently $\rho_0(i)$ and $\rho_1(i)$ must differ by at most $1 \pmod{k}$ as well. Moreover, since $k \geq 5$, $\rho_0(i)$ and $\rho_1(i)$ must differ by at most 1. Therefore, $\rho_0(m)$ and $\rho_1(m)$ are not merely both congruent to $0 \pmod{k}$, they are in fact equal. Inductively we have $\rho_0(m) = \rho_j(m), \;\; 0 \leq j \leq n$. Which leads to the following result:

$$w_f = \frac{\rho_0(m)}{k} = \frac{\rho_n(m)}{k} = w_g.$$

Thus w_f is independent of our choice of representative of $[f]$.

Step 4. Show that Φ is injective.

Suppose $\Phi(w) = [e]$, the equivalence class of the trivial map. The image of the trivial map is the single vertex 0, and it doesn't wind around C_k at all, so w must be 0.

This shows that Φ is an isomorphism, which completes this proof of the following theorem.

Theorem 2.6. (Malle [12], Barcelo et al. [3]) $A_1^G(C_k, 0) \simeq \mathbb{Z}$ *for* $k \geq 5$.

CHAPTER 3

ISOMORPHISMS OF G-HOMOTOPY GROUPS

3.1. Γ is the Box Product of Graphs

In the previous chapter, we recalled the definition of $A_1^G(\Gamma, v_0)$, the discrete fundamental group of a graph Γ, and computed this group for cycles. Our next step is to examine the structure of G-groups of more complicated graphs. When the G-homotopy relation on graph maps was defined, we first needed $\Gamma \Box \Gamma'$, the box product of Γ and Γ', and this seems like a natural place for us to continue our exploration. If we start with graphs Γ and Γ' for which we know the groups $A_1^G(\Gamma, v_0)$ and $A_1^G(\Gamma', v_0')$, and from these graphs we construct $\Gamma \Box \Gamma'$ with base vertex (v_0, v_0'), is there a relationship between $A_1^G(\Gamma \Box \Gamma', (v_0, v_0'))$ and the G-groups of Γ and Γ'?

Recall that each vertex in $\Gamma \Box \Gamma'$ is an ordered pair, (v, v'), where the first coordinate is a vertex in Γ and the second is a vertex in Γ'. Given a loop in $\Gamma \Box \Gamma'$ based at (v_0, v_0'), we can separate this sequence of ordered pairs into one sequence consisting of vertices in Γ, and a second sequence of vertices in Γ'. By the definition of edges in $\Gamma \Box \Gamma'$, one sees that these resulting sequences are loops in Γ and Γ', respectively. Furthermore, we can show that each equivalence class in $A_1^G(\Gamma \Box \Gamma', (v_0, v_0'))$ can be put in correspondence with a pair of equivalence classes, one from $A_1^G(\Gamma, v_0)$ and the other from $A_1^G(\Gamma', v_0')$. This leads us to the main result of this chapter, where we show that this correspondence is in fact a bijection from $A_1^G(\Gamma\Box\Gamma', (v_0, v_0'))$ to $A_1^G(\Gamma, v_0) \times A_1^G(\Gamma', v_0')$.

Theorem 3.1. *Let $\Gamma = (V, E)$, $\Gamma' = (V', E')$ be simple graphs with distinguished vertices, v_0 and v_0', respectively. Then $A_1^G(\Gamma\Box\Gamma', (v_0, v_0')) \simeq A_1^G(\Gamma, v_0) \times A_1^G(\Gamma', v_0')$.*

Proof. To prove this, we define a homomorphism $(\Phi_\Gamma, \Phi_{\Gamma'})$ from $A_1^G(\Gamma\Box\Gamma', (v_0, v_0'))$ to $A_1^G(\Gamma, v_0) \times A_1^G(\Gamma', v_0')$, then show that the homomorphism is invertible.

The homomorphism $(\boldsymbol{\Phi_\Gamma}, \boldsymbol{\Phi_{\Gamma'}})$. First, let p_Γ and $p_{\Gamma'}$ be the usual projection maps:

Given a graph map $f : (I_m, \partial I_m) \to (\Gamma\Box\Gamma', (v_0, v_0'))$, $p_\Gamma \circ f$ is a graph map which projects the image of f into Γ. From $p_\Gamma \circ f$, we induce the homomorphism $\Phi_\Gamma : A_1^G(\Gamma\Box\Gamma', (v_0, v_0')) \to A_1^G(\Gamma, v_0)$, defined by $\Phi_\Gamma([f]) = [p_\Gamma \circ f]$ [3]. Similarly, $\Phi_{\Gamma'}$ is a homomorphism from $A_1^G(\Gamma\Box\Gamma', (v_0, v_0'))$ to $A_1^G(\Gamma', v_0')$.

Since Φ_Γ and $\Phi_{\Gamma'}$ are homomorphisms, then $(\Phi_\Gamma, \Phi_{\Gamma'})$ is a homomorphism from $A_1^G(\Gamma\Box\Gamma', (v_0, v_0'))$ to $A_1^G(\Gamma, v_0) \times A_1^G(\Gamma', v_0')$. In fact, $(\Phi_\Gamma, \Phi_{\Gamma'})$ is an isomorphism. To show this, we define a map Ψ and show that it is the inverse of $(\Phi_\Gamma, \Phi_{\Gamma'})$.

Defining the homomorphism $\boldsymbol{\Psi}$. Let $[\alpha]$ be the equivalence class of $\alpha : (I_m, \partial I_m) \to (\Gamma, v_0)$ in $A_1^G(\Gamma, v_0)$, and let $[\beta]$ be the equivalence class of $\beta : (I_m, \partial I_m) \to (\Gamma', v_0')$ in $A_1^G(\Gamma', v_0')$. We want to define a map $\Psi : A_1^G(\Gamma, v_0) \times A_1^G(\Gamma', v_0') \to A_1^G(\Gamma\Box\Gamma', (v_0, v_0'))$ that maps the pair $([\alpha], [\beta])$ to an equivalence class in $A_1^G(\Gamma\Box\Gamma', (v_0, v_0'))$. Let $\Psi([\alpha], [\beta])$ be the equivalence class of the graph map $\psi_{\alpha,\beta} : (I_{2m}, \partial I_{2m}) \to (\Gamma\Box\Gamma', (v_0, v_0'))$ given by

$$\psi_{\alpha,\beta} = \begin{cases} (\alpha(i), v_0'), & 0 \leq i \leq m \\ \\ (v_0, \beta(i-m)), & m \leq i \leq 2m. \end{cases}$$

We must show that Ψ is well-defined. Let $\gamma : (I_m, \partial I_m) \to (\Gamma, v_0)$ be another representative of $[\alpha]$; let $\delta : (I_m, \partial I_m) \to (\Gamma', v_0')$ be another representative of $[\beta]$. We must show that $\psi_{\alpha,\beta}$ is G-homotopic to $\psi_{\gamma,\delta}$. Let F be a G-homotopy from α to γ and let F' be a G-homotopy from β to δ. If necessary, we may stretch one of the G-homotopies as we did in the proof of Proposition 2.3 so that both F and F' are defined on $I_m \Box I_n$. We now define a new map and show that it is the desired G-homotopy from $\psi_{\alpha,\beta}$ to $\psi_{\gamma,\delta}$.

Define $F'' : I_{2m} \Box I_n \to (\Gamma \Box \Gamma', (v_0, v_0'))$ by

$$F''(i,j) = \begin{cases} (F(i,j), v_0'), & 0 \leq i \leq m, \quad 0 \leq j \leq n \\ \\ (v_0, F'(i-m, j)), & m \leq i \leq 2m, \quad 0 \leq j \leq n \end{cases}$$

We can verify that F'' is a graph map that satisfies:

1. $F''(i,0) = \begin{cases} (F(i,0), v_0') = (\alpha(i), v_0'), & 0 \leq i \leq m \\ \\ (v_0, F'(i-m, j)) = (v_0, \beta(i-m)), & m \leq i \leq 2m \end{cases}$

2. $F''(i,n) = \begin{cases} (F(i,n), v_0') = (\gamma(i), v_0'), & 0 \leq i \leq m \\ \\ (v_0, F'(i-m, j)) = (v_0, \delta(i-m)), & m \leq i \leq 2m \end{cases}$

3. $F''(0,j) = (F(0,j), v_0') = (\alpha(0), v_0') = (v_0, v_0'), \quad 0 \leq j \leq n$

4. $F''(2m,j) = (v_0, F'(2m,j)) = (v_0, \beta(m)) = (v_0, v_0'), \quad 0 \leq j \leq n.$

Consequently F'' is a G-homotopy from $\psi_{\alpha,\beta}$ to $\psi_{\gamma,\delta}$, and therefore Ψ is well-defined.

We can describe the G-homotopy grid for F'' in terms of the grids for $F : I_m \Box I_n \to (\Gamma, v_0)$ and $F' : I_m \Box I_n \to (\Gamma', v_0')$. First, add v_0' to the image of each vertex in the grid for

F as the second coordinate, so that each vertex in the grid is now labeled with a vertex in $\Gamma\Box\Gamma'$; clearly this preserves adjacency. Similarly, add v_0 so that is is now the first coordinate in the image of each vertex in the grid for F'; again, each vertex in the grid is now labeled with a vertex in $\Gamma\Box\Gamma'$. In particular, each vertex in the last column of the grid for F, as well as each vertex in the first column of the grid for F', is labeled with (v_0, v_0'), so we may paste the grids together along these columns. The result is the G-homotopy grid for $F'' : I_{2m}\Box I_n \to (\Gamma\Box\Gamma', (v_0, v_0'))$

The homomorphism $\boldsymbol{\Psi}$ is the inverse of $(\boldsymbol{\Phi_{\Gamma}}, \boldsymbol{\Phi_{\Gamma'}})$. The composition $(\Phi_\Gamma, \Phi_{\Gamma'}) \circ \Psi$ is straightforward:

$$\begin{aligned} ((\Phi_\Gamma, \Phi_{\Gamma'}) \circ \Psi)([\alpha], [\beta]) &= ((\Phi_\Gamma \circ \Psi)([\alpha], [\beta]), (\Phi_{\Gamma'} \circ \Psi)([\alpha], [\beta])) \\ &= ([p_\Gamma \circ \psi_{\alpha,\beta}], [p_{\Gamma'} \circ \psi_{\alpha,\beta}]). \end{aligned}$$

When p_Γ projects the image of $\psi_{\alpha,\beta}$ into Γ, we get a new graph map $(I_{2m}, \partial I_{2m}) \to (\Gamma, v_0)$ given by

$$p_\Gamma \circ \psi_{\alpha,\beta}(i) = \begin{cases} \alpha(i) & 0 \le i \le m \\ v_0 & m \le i \le 2m. \end{cases}$$

This is simply the map α, stretched so that it is defined on I_{2m}. Similarly, $p_{\Gamma'} \circ \psi_{\alpha,\beta}$ is defined by

$$p_{\Gamma'} \circ \psi_{\alpha,\beta}(i) = \begin{cases} v_0' & 0 \le i \le m \\ \beta(i-m) & m \le i \le 2m. \end{cases}$$

Therefore, $((\Phi_\Gamma, \Phi_{\Gamma'}) \circ \Psi)([\alpha], [\beta]) = ([\alpha], [\beta])$ as desired.

The composition $\Psi \circ (\Phi_\Gamma, \Phi_{\Gamma'})$ is somewhat more complex:

$$\begin{aligned}(\Psi \circ (\Phi_\Gamma, \Phi_{\Gamma'}))([f]) &= \Psi(\Phi_\Gamma([f]), \Phi_{\Gamma'}([f])) \\ &= \Psi([p_\Gamma \circ f], [p_{\Gamma'} \circ f]).\end{aligned}$$

The image of f is a loop in $\Gamma \Box \Gamma'$ based at (v_0, v_0'). The map p_Γ projects this loop into Γ, thus the image of $p_\Gamma \circ f$ is a loop in Γ based at v_0. Similarly, the image of $p_{\Gamma'} \circ f$ is a loop in Γ' based at v_0'. Therefore, the image of $\psi_{p_\Gamma \circ f, p_{\Gamma'} \circ f}$ is the concatenation of two loops in $\Gamma \Box \Gamma'$ based at (v_0, v_0'). In the first loop, the set of first coordinates of the vertices correspond to the image of $p_\Gamma \circ f$, and the second coordinate of each vertex is v_0'. In the second loop, the first coordinate of each vertex is v_0, and the set of second coordinates corresponds to the image of $p_{\Gamma'} \circ f$. However, it is not immediately clear that this concatenation of loops in $\Gamma \Box \Gamma'$ is G-homotopic to the original loop for f. To prove that this is in fact the case, we define a map $G : I_{2m} \Box I_m \to \Gamma \Box \Gamma'$ and show that G is a G-homotopy from $\psi_{p_\Gamma \circ f, p_{\Gamma'} \circ f}$ to f.

Let $G : I_{2m} \Box I_m \to \Gamma \Box \Gamma'$ be defined as follows:

The bottom row in the G-homotopy grid for G corresponds to $\psi_{p_\Gamma \circ f, p_{\Gamma'} \circ f}$, the concatenation of the two loops in $\Gamma \Box \Gamma'$. For ease of notation, we refer to this as row 0, to coincide with the second coordinate of the vertices in the bottom row of $I_{2m} \Box I_m$.

$$G(i, 0) = \begin{cases} (p_\Gamma(f(i)), v_0'), & 0 \le i \le m \\ (v_0, p_{\Gamma'}(f(i - m))), & m \le i \le 2m. \end{cases}$$

In row 1, we replace v_0' with $p_{\Gamma'}(f(1))$ for $1 \le i \le m$. In this and all subsequent rows, we leave the first coordinate of each vertex unchanged.

$$G(i,1) = \begin{cases} (p_\Gamma(f(i)), p_{\Gamma'}(f(i))), & i = 0 \\ (p_\Gamma(f(i)), p_{\Gamma'}(f(1))), & 1 \le i \le m \\ (v_0, p_{\Gamma'}(f(i-m))), & m+1 \le i \le 2m. \end{cases}$$

In row 2, we replace $p_{\Gamma'}(f(1))$ with $p_{\Gamma'}(f(2))$ for $2 \le i \le m+1$.

$$G(i,2) = \begin{cases} (p_\Gamma(f(i)), p_{\Gamma'}(f(i))), & 0 \le i \le 1 \\ (p_\Gamma(f(i)), p_{\Gamma'}(f(2))), & 2 \le i \le m \\ (v_0, p_{\Gamma'}(f(2))), & i = m+1 \\ (v_0, p_{\Gamma'}(f(i-m))), & m+2 \le i \le 2m. \end{cases}$$

For each remaining row of G, for $3 \le j \le m$, we continue to replace $p_{\Gamma'}(f(j-1))$ with $p_{\Gamma'}(f(j))$ for $j \le i \le m+j-1$.

$$G(i,j) = \begin{cases} (p_\Gamma(f(i)), p_{\Gamma'}(f(i))), & 0 \le i \le j-1 \\ (p_\Gamma(f(i)), p_{\Gamma'}(f(j))), & j \le i \le m \\ (v_0, p_{\Gamma'}(f(j))), & m+1 \le i \le m+j-1 \\ (v_0, p_{\Gamma'}(f(i-m))), & m+j \le i \le 2m. \end{cases}$$

Note that in the last row of the grid, with $j = m$, this results in

$$G(i,m) = \begin{cases} (p_\Gamma(f(i)), p_{\Gamma'}(f(i))), & 0 \le i \le m \\ (v_0, p_{\Gamma'}(f(m))), & m+1 \le i \le 2m \end{cases}$$

$$= \begin{cases} (p_\Gamma(f(i)), p_{\Gamma'}(f(i))), & 0 \le i \le m \\ (v_0, v_0'), & m+1 \le i \le 2m. \end{cases}$$

$$= \begin{cases} f(i), & 0 \le i \le m \\ (v_0, v_0'), & m < i \le 2m. \end{cases}$$

Thus we have

1. $G(i,0) = \psi_{p_\Gamma \circ f,\, p_{\Gamma'} \circ f}(i)$ for $0 \le i \le 2m$.
2. $G(i,m) = f(i)$ for $0 \le i \le 2m$.
3. $G(0,j) = G(2m,j) = (v_0, v_0')$ for $0 \le j \le n$.

We still need to show that G preserves adjacency to conclude that it is a G-homotopy from $\psi_{p_\Gamma \circ f,\, p_{\Gamma'} \circ f}$ to f, but first we will illustrate the construction of G with an example.

Example. We use $C_5 \Box C_5$ as an example. To avoid the confusion of referring to two copies of the same graph with the same vertex labels, we refer to the first copy of C_5 simply as C, with vertices $\{0,1,2,3,4\}$, and to the second copy as C', with vertices $\{0',1',2',3',4'\}$. Define $f : (I_{10}, \partial I_{10}) \to (C \Box C', (0,0'))$ as shown in Figure 6.

One verifies easily that f is a graph map and its image is a loop in $C \Box C'$ based at $(0,0')$. The image of $p_C \circ f$ is the loop $0-0-0-0-1-2-3-3-4-0-0$ in C and the image of $p_{C'} \circ f$ is the loop $0'-1'-2'-3'-3'-3'-3'-4'-4'-4'-0'$ in C'. The

Figure 6. $f : (I_{10}, \partial I_{10}) \to (C\Box C', (0, 0'))$

image of $\psi_{p_C \circ f, p_{C'} \circ f}$ is the loop of 21 vertices in $C\Box C'$: $(0,0') - (0,0') - (0,0') - (0,0') - (1,0') - (2,0') - (3,0') - (3,0') - (4,0') - (0,0') - (0,0') - (0,1') - (0,2') - (0,3') - (0,3') - (0,3') - (0,3') - (0,4') - (0,4') - (0,4') - (0,0')$.

baselineskip

The grid in Figure 7 illustrates the G-homotopy $G : (I_{20}\Box I_{10}) \to (C\Box C', (0,0'))$ from $\psi_{p_C \circ f, p_{C'} \circ f}$ to f. The bottom row of the grid, row 0, is labeled with the image just described, and the top row is labeled with the image of f, extended by adding additional copies of the base vertex $(0, 0')$. The central section of the grid, where portions of the rows have been underlined, is where changes are made from row to row. In row 1, when we replace the second coordinate, $0'$, with $p_{c'}(f(1)) = 1'$ for $1 \leq i \leq 10$, we refer to this as *sliding* the label $1'$ to the left across the row, to all but the last vertex in the row. In the next row, we slide the label $2'$ to the left, changing the second coordinate in all but the last two vertices. We continue this sliding process for each of the following rows in the grid, sliding the appropriate label to the left across $m = 10$ vertices in the row. In each consecutive row, we are reconstructing the image of our original graph map f one vertex at a time, which is then followed by m copies of the base vertex $(0, 0')$ in the last row.

The sliding process preserves adjacency. All that remains is to show that G is a graph map by verifying that it preserves adjacency. First, we can consider the horizontal edges in the grid connecting vertices (i, j) and $(i + 1, j)$ according to three possible cases.

Case 1. $i \leq j - 1$. This corresponds to the horizontal edges in the upper left section of the grid. These are the vertices where we have reconstructed the image of f, so adjacency is preserved because we assumed f is a graph map.

<u>**Case 2.**</u> $j \leq i \leq m+j-1$. These are the horizontal edges in the central section of the grid where the sliding occurs. In a single row, the first coordinates have remained unchanged from the first row, so they correspond to vertices that are adjacent or identical in C. The second coordinates are identical as a result of the sliding, so adjacency is preserved in $C\Box C'$ as well.

<u>**Case 3.**</u> $i \geq m+j$. This is the lower right portion of the grid, where no changes have been made from the first row.

Second, for a fixed column in the grid, we left the first coordinate of each vertex unchanged, so we only need to consider the second coordinates of $G(i,j)$ and $G(i,j+1)$. If the second coordinates are not identical, then in row j the second coordinate is $p_{C'}(f(j))$ and in row $j+1$ the second coordinate is $p_{C'}(f(j+1))$, because this is the only change made from row to row. These coordinates are adjacent or identical in C', so $G(i,j)$ and $G(i,j+1)$ are adjacent or identical in $C\Box C'$. Therefore all edges in the A-homotopy grid are preserved by G so indeed G is a G-homotopy from $\psi_{p_C \circ f, p_{C'} \circ f}$ to f.

This sliding technique does not necessarily result in the shortest G-homotopy possible between two maps: notice that the technique made no changes in rows 4, 5, and 6 in Figure 7 because there are repeated vertices in the image of $p_{C'} \circ f$. However, it always results in a valid G-homotopy. This demonstrates that $\Psi \circ (\Phi_\Gamma, \Phi_{\Gamma'})([f]) = [f]$ and that Ψ is indeed the inverse of $(\Phi_\Gamma, \Phi_{\Gamma'})$. Therefore $(\Phi_\Gamma, \Phi_{\Gamma'})$ is the desired isomorphism from $A_1^G(\Gamma\Box\Gamma', (v_0, v_0'))$ to $A_1^G(\Gamma, v_0) \times A_1^G(\Gamma', v_0')$. $\Box$

In addition, the map $f : \Gamma\Box\Gamma' \to \Gamma'\Box\Gamma$ that sends (v, v') to (v', v) is a bijection that preserves adjacency, so the operation of constructing the box product of two graphs is commutative, which gives us the following isomorphism.

Proposition 3.2. *Let $\Gamma = (V, E)$, $\Gamma' = (V', E')$ be simple graphs without loops and with distinguished vertices, v_0 and v_0', respectively. Then $A_1^G(\Gamma \square \Gamma', (v_0, v_0')) \simeq A_1^G(\Gamma' \square \Gamma, (v_0', v_0))$.*

3.2. Γ Has a Cut Set of Size 1 or 2

If a graph Γ is not the box product of two or more smaller graphs, we can look for other ways to express Γ as being constructed from smaller graphs, and then try to use the G-groups of the smaller graphs to help us describe the G-group of the larger graph. We consider the cases where Γ is a connected graph and has a cut set of size 1 or 2, and to do so we make use of the following Seifert-Van Kampen type theorem from Barcelo et al. Note that in general we now refer to loops or cycles in a graph Γ, the images of based graph maps from I to Γ, rather than the graph maps themselves.

Theorem 3.3. [3] *Let Γ be a connected, simple graph and let v be a vertex in Γ. Let Γ_1 and Γ_2 be two connected subgraphs of Γ such that $\Gamma = \Gamma_1 \cup \Gamma_2$, and $\Gamma_1 \cap \Gamma_2$ is connected and contains v. Suppose further that any 3-cycles or 4-cycles in Γ that contain v are completely contained in either Γ_1 or Γ_2; that is, we do not "break" any 3-cycles or 4-cycles when we separate Γ into Γ_1 and Γ_2. Then*

$$A_1^G(\Gamma, v) \simeq (A_1^G(\Gamma_1, v) * A_1^G(\Gamma_2, v))/V$$

where V is the normal subgroup of the free product generated by all elements of the form $[l] * [l]^{-1}$ for a loop l in $\Gamma_1 \cap \Gamma_2$ based at v.

If we have a connected graph Γ with a cut vertex, v, we first remove v to break Γ into components $\Gamma_1^*, \Gamma_2^*, \ldots, \Gamma_k^*$. For each of these components, we then replace v and all edges incident to it, resulting in a collection of connected subgraphs, $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$, such that Γ is the union of these subgraphs, and the pairwise intersection of the subgraphs is the single vertex v. Consequently, any loop l contained in the intersection must be the trivial loop consisting solely of v, so it is easy to see that the normal subgroup V defined in Theorem 3.3 is trivial. We may therefore use Theorem 3.3 and apply induction on k to see that the discrete fundamental group of Γ is isomorphic to the free product of the discrete fundamental groups of each of the subgraphs, resulting in the following corollary.

Corollary 3.4. *Let Γ be a connected, simple graph with a cut vertex, v. Let $\Gamma_1^*, \Gamma_2^*, \ldots, \Gamma_k^*$ be the components of $\Gamma - v$. Let $\Gamma_i = \Gamma[\Gamma_i^* \cup v]$, $1 \leq i \leq k$, be the induced subgraph on v and the vertices of Γ_i^*. Then $A_1^G(\Gamma, v) \simeq A_1^G(\Gamma_1, v) * A_1^G(\Gamma_2, v) * \cdots * A_1^G(\Gamma_k, v)$.*

A *minimal cut set* of a connected graph Γ is a subset $S \subset V$ of smallest cardinality such $\Gamma - S$ is disconnected. Suppose that Γ has a minimal cut set of size two, $\{v, w\}$. The distance beween v and w, $d(v, w)$, is the length of the shortest path from v to w in Γ. We consider the cases where $d(v, w) \leq 2$ and $d(v, w) \geq 3$ separately.

If $d(v, w) \leq 2$, then we can use a similar approach to that in the proof of Corollary 3.4. We define $\Gamma_1^*, \Gamma_2^*, \ldots, \Gamma_k^*$ to be the components of $\Gamma - \{v, w\}$, and rather than replacing a single vertex in each of these components, we want to create a connected subgraph of Γ that we can add to each component and that is the pairwise intersection of the resulting subgraphs. Here, we use $N(v, w)$, the *common neighborhood of v and w*; that is, v, w, and all vertices adjacent to both v and w. The induced subgraph on these vertices, $\Gamma[N(v, w)]$, is connected. Let $\Gamma_i = \Gamma[\Gamma_i^* \cup N(v, w)]$, $1 \leq i \leq k$, be the induced subgraph on the vertices of Γ_i^* and $N(v, w)$. The set $\{v, w\}$ is a minimal cut set, so v and w must each have at least one neighbor in Γ_i^*, otherwise we could disconnect the vertices in Γ_i^* from the rest of the graph by removing only one vertex in $\{v, w\}$. Therefore each of the Γ_i is connected. Furthermore, $\Gamma_i \cap \Gamma_j$ for $i \neq j$ is precisely $\Gamma[N(v, w)]$. Without loss of generality, we may let v be the base vertex of Γ and each of the Γ_i.

Theorem 3.5. *Let Γ be a connected, simple graph with a minimal cut set $\{v, w\}$ such that $d(v, w) \leq 2$. Let $\Gamma_1^*, \Gamma_2^*, \ldots, \Gamma_k^*$ be the components of $\Gamma - \{v, w\}$. Let $\Gamma_i = \Gamma[\Gamma_i^* \cup N(v, w)]$, $1 \leq i \leq k$, be the induced subgraph on the vertices of Γ_i^* and $N(v, w)$. Then $A_1^G(\Gamma, v) \simeq A_1^G(\Gamma_1, v) * A_1^G(\Gamma_2, v) * \cdots * A_1^G(\Gamma_k, v)$.*

Proof. We must first show that any 3-cycles or 4-cycles based at v are completely contained in one of the Γ_i. Suppose $C = vv_1v_2v_3$ is a 4-cycle based at v.

Case 1. $\mathbf{w = v_1}$ or $\mathbf{v_3}$. Without loss of generality let $w = v_1$. If both v_2 and v_3 are in $N(v,w)$, then C is completely contained in the intersection of the Γ_i. If only one vertex in $\{v_2, v_3\}$ is in $N(v,w)$, and the other is in one of the Γ_i^*, then C is contained in Γ_i. If neither v_2 nor v_3 are in $N(v,w)$, then they must both be in the same Γ_i^* because they are adjacent, so again C is contained in Γ_i.

Case 2. $\mathbf{w = v_2}$. Then both v_1 and v_3 are in $N(v,w)$, and C is contained in the intersection of the subgraphs.

Case 3. $\mathbf{w \notin C}$. Could we have two vertices in the cycle that lie in different components of $\Gamma - \{v,w\}$? Any path in Γ with end vertices in different components of $\Gamma - \{v,w\}$ must contain an interior vertex from $\{v,w\}$ so that the endpoints are disconnected when the cut set is removed. However, $v_1v_2v_3$ is a path in Γ that is not be affected by the removal of v and w, so all three vertices in the path must be in the same component Γ_i^*, and consequently C is contained in Γ_i.

If C is a 3-cycle based at v and $w \in C$, then the third vertex is in $N(v,w)$, and C is contained in the intersection of the subgraphs. If $w \notin C$, then the remaining two vertices are adjacent and therefore must be in the same component Γ_i^* of $\Gamma - \{v,w\}$, and C must be contained in Γ_i. Consequently, no 3-cycles or 4-cycles are broken when we remove v and w from Γ.

Now let C be a cycle of length ≥ 5 in $\Gamma[N(v,w)] = \bigcap_{i=1}^{k} \Gamma_i$. Figure 10 illustrates the possible cases depending on whether C contains one or more vertices of $\{v,w\}$. It is easy to see that since both v and w are adjacent to all of the other vertices in $N(v,w)$, the cycle C is the concatenation of 3-cycles, and possibly one 4-cycle if both v and w are in C. Clearly any loop in $\Gamma[N(v,w)]$ based at v is G-homotopic to the trivial loop, so the normal subgroup V defined in Theorem 3.3 is trivial. We may therefore apply Theorem 3.3 and induction on the number of subgraphs to complete the proof of Theorem 3.5. □

The last class of graphs that we examine also have a minimal cut set $\{v, w\}$, but with $d(v, w) \geq 3$. In this case we define a collection of connected subgraphs of Γ, each containing the base vertex v, and show that each loop in Γ based at v can be expressed as the concatenation of loops from the subgraphs, and consequently the discrete fundamental group of Γ is isomorphic to the free product of the discrete fundamental groups of the individual subgraphs.

Theorem 3.6. *Let Γ be a connected, simple graph with a minimal cut set $\{v, w\}$ such that $d(v, w) \geq 3$. Let $\Gamma_1^*, \Gamma_2^*, \ldots, \Gamma_k^*$ be the components of $\Gamma - \{v, w\}$. Let $\Gamma_i = \Gamma[\Gamma_i^* \cup \{v, w\}]$. Let P_i be a path from v to w in Γ_i. Let Γ' be the graph consisting of the set of P_i, $1 \leq i \leq k$. Then $A_1^G(\Gamma, v) = A_1^G(\Gamma_1, v) * A_1^G(\Gamma_2, v) * \cdots * A_1^G(\Gamma_k, v) * A_1^G(\Gamma', v)$.*

Proof. We note that Γ', the union of the k $v{-}w$ paths, is connected. Clearly, a concatenation of loops in the collection of subgraphs $\left(\bigcup_{i=1}^{k} \Gamma_i\right) \cup \Gamma'$, each based at v, is a loop in Γ based at v. Thus the product of the equivalence classes of the loops corresponds to the equivalence class of the loop in Γ. We must also show that a loop in Γ based at v is G-homotopic to the concatenation of a collection of loops from the subgraphs of Γ defined in the statement of the theorem. Let L be a loop in Γ based at v. If L is contained in one of the Γ_i or Γ', then we are done. If L is not completely contained in one of the subgraphs, then we can break L into the concatenation of strings such that each string has endpoints in $\{v, w\}$ and is contained in one of the Γ_i. The *intermediate endpoints of L* are the endpoints of the strings, except for the first endpoint of the first string and last endpoint of the last string, which are both v.

$$L = v \underbrace{\cdots\cdots}_{\Gamma_{i_1}} w \underbrace{\cdots\cdots}_{\Gamma_{i_2}} w \underbrace{\cdots\cdots}_{\Gamma_{i_3}} v \underbrace{\cdots\cdots}_{\Gamma_{i_4}} w \underbrace{\cdots\cdots}_{\Gamma_{i_5}} v$$

Consider the intermediate endpoints of L. If an intermediate endpoint is a w, and it connects a string in Γ_{i_j} to a string in $\Gamma_{i_{j+1}}$, replace w with $P_{i_j}^{-1}P_{i_j}P_{i_{j+1}}^{-1}P_{i_{j+1}}$. If the strings connected by w are in the same subgraph of Γ, or if the endpoint is v, no replacement is necessary. Inserting $P_{i_j}^{-1}$ extends the string before the endpoint w by adding a string ending at v, turning it into a loop in Γ_{i_j} based at v. The two paths $P_{i_j}P_{i_{j+1}}^{-1}$ form a loop in Γ'. Inserting $P_{i_{j+1}}$ adds a path to the beginning of the string after the w so that it now begins at v; the string also ends at v after all changes are made, yielding a loop in $\Gamma_{i_{j+1}}$ based at v. The resulting loop, L', is the concatenation of loops in the collection of Γ_i, combined with loops from Γ'. Each replacement of four paths, $P_{i_j}^{-1}P_{i_j}P_{i_{j+1}}^{-1}P_{i_{j+1}}$, that we made is G-homotopic to the trivial loop, so $L \simeq_G L'$ as desired, and the equivalence class in $A_1^G(\Gamma, v)$ containing L is the product of the equivalence classes of the smaller loops.

□

Example. Let Γ be the graph shown in Figure 11, with minimal cut set $\{v, w\}$. There are three components in $\Gamma - \{v, w\}$, so Γ' is the union of one $v - w$ path from each of the three subgraphs, Γ_1, Γ_2, and Γ_3. Let L be the loop shown in Figure 11. We can break L into five strings with endpoints in $\{v, w\}$ and express it as

$$L = S_1S_2S_3S_4S_5.$$

The first intermediate endpoint, w, connects S_1 in Γ_1 and S_2 in Γ_3, so we replace w with $P_1^{-1}P_1P_3^{-1}P_3$. We then replace the second w endpoint with $P_3^{-1}P_3P_2^{-1}P_2$. The third intermediate endpoint, connecting S_3 and S_4, is a v, so we do not make any replacement.

Finally, we replace the last w endpoint with $P_3^{-1}P_3P_1^{-1}P_1$. The resulting loop is

$$L' = \underbrace{S_1P_1^{-1}}_{\text{loop in }\Gamma_1}\,\underbrace{P_1P_3^{-1}}_{\Gamma'}\underbrace{P_3S_2P_3^{-1}}_{\Gamma_3}\underbrace{P_3P_2^{-1}}_{\Gamma'}\underbrace{P_2S_3}_{\Gamma_2}\underbrace{S_4P_3^{-1}}_{\Gamma_3}\underbrace{P_3P_1^{-1}}_{\Gamma'}\underbrace{P_1S_5}_{\Gamma_1}$$

which is the concatenation of loops in Γ' and the collection of subgraphs Γ_i, $1 \leq i \leq 3$.

Currently, it does not seem that we may easily generalize the techniques used in the proofs of Theorems 3.5 and 3.6 if a minimal cut set in Γ contains three or more vertices. For the present, therefore, we set aside graphs with larger minimal cut sets for possible future consideration, and in the remaining chapters we explore the G-groups of graphs related to finite graded lattices.

Figure 7. A G-homotopy diagram showing the sliding technique.

Figure 8. Γ has a cut vertex, v.

Figure 9. Subgraphs of Γ where $d(v,w) \leq 2$.

Figure 10. Cycles in $\Gamma[N(v,w)]$ are G-homotopic to a single vertex.

Figure 11. Three $v - w$ paths that comprise Γ', and a loop in Γ based at v.

CHAPTER 4

NEW LATTICES FROM OLD

4.1. Graphs Arising from the Order Complex of a Lattice

In the previous chapter, we proved results related to the G-groups of graphs that have minimal cut sets of size one or two, or can be viewed as the box product of two graphs. In this and the following chapters, we consider graphs arising from a particular simplicial complex, the order complex of a lattice. We use these graphs to obtain results for the discrete fundamental groups of the order complexes. In the development of A-theory, Barcelo et al. first defined a discrete homotopy theory for simplicial complexes. Let Δ be a simplicial complex of dimension d, let $0 \leq q \leq d$ be fixed, and let σ_0 be a given maximal simplex of dimension at least q.

Definition 4.1. (Barcelo et al. [3])

1. *Two simplices σ and τ in Δ are q-connected if there is a sequence of simplices $\sigma, \sigma_1, \sigma_2, \ldots, \sigma_n, \tau$ such that any two consecutive simplices share a q-face, that is, they have at least $q+1$ vertices in common. Such a sequence is called a q-chain, and its length is n.*

2. *A q-loop in Δ based at σ_0 is a q-chain beginning and ending at σ_0. Denote a q-loop $\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_n, \sigma_0$ by $(\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_n, \sigma_0) = (\sigma)$.*

This definition of a q-loop reminds us of the definition of a graph map, and the equivalence relation on these loops, A-homotopy, is equally familiar.

Definition 4.2. [3] *Let $\simeq_A$ be the equivalence on the collection of q-loops in Δ, based at σ_0, generated by the following two conditions.*

1. *The q-loop*

$$(\sigma) = (\sigma_0, \ldots, \sigma_i, \sigma_{i+1}, \ldots, \sigma_n, \sigma_0)$$

   is equivalent to the q-loop

$$(\sigma') = (\sigma_0, \ldots, \sigma_i, \sigma_i, \sigma_{i+1}, \ldots, \sigma_n, \sigma_0).$$

   As with graph maps, we can stretch a q-loop by repeating simplices.

2. *Suppose that (σ) and (τ) have the same length. They are equivalent if there is a grid, analogous to a G-homotopy grid, where the vertices in the first row of the grid correspond to the simplices of (σ), the vertices in the last row correspond to the simplices of (τ), and each horizontal or vertical edge between simplices in the grid indicates that they share a q-face.*

The discrete fundamental group of Δ, denoted by $A_1^q(\Delta, \sigma_0)$, is the set of A-homotopy equivalence classes of q-loops based at σ_0, with a product operation defined by concatenation of loops. This is also referred to as the A_1-group of Δ. Kramer and Laubenbacher [11] showed that to compute this group, one needed to build a graph $\Gamma^q(\Delta)$ with vertices corresponding to simplices of Δ of dimension $\geq q$ and each edge corresponding to a pair of simplices that share a q-face. Later on, Barcelo et al. [3] showed that $A_1^q(\Delta, \sigma_0) \simeq A_1^G(\Gamma^q(\Delta), v_0)$, where v_0 is the vertex corresponding to maximal simplex σ_0. In addition, they showed that we can use the potentially much smaller subgraph $\Gamma^q_{max}(\Delta)$, where the vertices correspond to maximal simplices of dimension $\geq q$, and they also proved that $A_1^q(\Delta, \sigma_0) \simeq A_1^G(\Gamma_{max}^q(\Delta), v_0)$.

In this chapter, we consider one particular type of simplicial complex, the order complex of a truncated finite graded lattice, denoted by $\Delta(\overline{L})$. One of the first simplicial complexes for which the A_1-group was computed was the barycentric subdivision of the boundary of the $(n-1)$-simplex, which corresponds to $\Delta(\overline{B_n})$, the order complex of the Boolean lattice of rank n with the minimal and maximal elements removed. Even though the computer program $A1$ [9] that we have for computing the abelianization of A_1-groups is quite efficient, the trouble lies in the complexity of the order complex associated to a lattice, and values for $n \geq 8$ were impossible to compute. On the other hand, B_n is easily seen to be isomorphic to $\mathbf{2}^n$, the direct product of n copies of the poset $\mathbf{2}$ on two elements, x and y, where $x < y$. Therefore, driven by these considerations and by our theorem on the product of graphs, we set out to look for a way to compute $A_1^{n-3}(\Delta(\overline{B_n}))$ for any n. To begin, we look at the order complex of a lattice. In this chapter, we adopt Stanley's [14] conventions and definitions for lattices.

Let L be a finite *graded lattice of rank* k, a lattice in which every maximal chain is of length k [14]. Recall that the simplicial complex $\Delta(\overline{L})$, called the *order complex* of $\overline{L}$, is defined in [5] as follows: the ground set of $\Delta(\overline{L})$ is the set of elements of $\overline{L} = L - \{\hat{0}, \hat{1}\}$, and the i-faces of $\Delta(\overline{L})$ are the i-chains, $x_0 < x_1 < x_2 < \cdots < x_i$, of $\overline{L}$. Thus $\Delta(\overline{L})$ is a pure simplicial complex of dimension $k-2$. The vertices of the graph $\Gamma_{max}^{k-3}(\Delta(\overline{L}))$ correspond to the maximal simplices of $\Delta(\overline{L})$, and consequently to the maximal chains in $\overline{L}$, or for that matter in L as well. Two simplices in $\Gamma_{max}^{k-3}(\Delta(\overline{L}))$ are adjacent if they share a $(k-3)$-face, or, equivalently, if the corresponding maximal chains, $\overline{C}$ and $\overline{C'}$, differ in precisely one element at rank i for some $1 \leq i \leq k-1$. In this case, we say that $\overline{C}$ and $\overline{C'}$ are *adjacent* chains, denoted by $\overline{C} \sim \overline{C'}$.

The resulting graph $\Gamma^{k-3}_{max}(\Delta(\overline{L}))$ is then used to compute $A^{k-3}_1(\Delta(\overline{L}), \overline{C})$, the discrete fundamental group of $\Delta(\overline{L})$ based at the vertex corresponding to a maximal chain $\overline{C}$ in $\overline{L}$. Recall that if a graph is connected, then the G-group of the graph is independent of the choice of base vertex. However, we note that the graph $\Gamma^{k-3}_{max}(\Delta(\overline{L}))$ need not be connected, and therefore we need to identify the base vertex when referring to the G-group. Furthermore, we note that in a lattice L of rank ≥ 2, there is a one-to-one correspondence between maximal chains in L and in $\overline{L}$. For the sake of consistency, if L is a lattice of rank 1, we define $\Gamma^{-2}_{max}(\Delta(\overline{L}))$ to be a single vertex corresponding to the empty chain. Therefore, $\Gamma^{k-1}_{max}(\Delta(L)) \simeq \Gamma^{k-3}_{max}(\Delta(\overline{L}))$.

4.2. Sums of Lattices

Given two graded lattices, L_1 and L_2, of ranks k and l, respectively, we can use these lattices to construct new lattices in a variety of ways, see for example [14]. We assume that the elements of L_1 and L_2 are disjoint sets. We consider the structure of these new lattices and their associated graphs, in the hope of finding a relationship between the discrete fundamental groups of the order complexes of the individual lattices and of the new lattices.

The elements of our first new poset, $L_1 + L_2$, the *direct sum of* L_1 *and* L_2, are the elements in $L_1 \cup L_2$ with the partial order given by $x \leq y$ in $L_1 + L_2$ if either $x, y \in L_1$ and $x \leq y$ in L_1, or $x, y \in L_2$ and $x \leq y$ in L_2. The direct sum is a poset but not a lattice, because it contains neither maximal nor minimal element. We can add elements $\hat{0}$ and $\hat{1}$ to the poset to obtain a lattice that is denoted as $\widehat{L_1 + L_2}$. For our purposes, we also assume that L_1 and L_2 are both of rank k, ensuring that $\widehat{L_1 + L_2}$ is a graded lattice, of rank $k+2$. A maximal chain in $\widehat{L_1 + L_2}$ corresponds to a maximal chain in either L_1 or L_2 (with the $\hat{0}$ and $\hat{1}$ added). Two maximal chains $\overline{C}_1$ and $\overline{C}_2$ in $L_1 + L_2$ (note that this is the truncation of $\widehat{L_1 + L_2}$), which correspond to maximal chains in L_1 and L_2, respectively, can differ in precisely one element if and only if L_1 and L_2 each consist of a single element. Therefore $\Gamma^{k-1}_{max}(\Delta(L_1+L_2))$, the graph associated to the order complex of L_1+L_2, is connected if and only if L_1 and L_2 are both single elements. If $k > 0$, then $\Gamma^{k-1}_{max}(\Delta(L_1 + L_2))$ is simply the disjoint union of $\Gamma^{k-3}_{max}(\Delta(\overline{L_1}))$ and $\Gamma^{k-3}_{max}(\Delta(\overline{L_2}))$. Consequently, the discrete fundamental group depends solely on our selection of the base vertex in $\Gamma^{k-1}_{max}(\Delta(L_1 + L_2))$, resulting in the following theorem.

Theorem 4.3. *Let L_1 and L_2 be two finite graded lattices, both of rank $k > 0$. Let C be a maximal chain in $L_1 + L_2$. Let $\overline{C}$ be the corresponding maximal chain in either $\overline{L}_1$ or $\overline{L}_2$. Then*

$$A_1^{k-1}(\Delta(L_1 + L_2), C) \simeq \begin{cases} A_1^{k-3}(\Delta(\overline{L}_1), \overline{C}) & \text{if } C \in L_1 \\ \\ A_1^{k-3}(\Delta(\overline{L}_2), \overline{C}) & \text{if } C \in L_2. \end{cases}$$

Figure 12. The direct sum of two graded lattices of the same rank.

The elements of $L_1 \oplus L_2$, the *ordinal sum of* L_1 *and* L_2, are the elements in $L_1 \cup L_2$, with $x \leq y$ in $L_1 \oplus L_2$ if

1. $x, y \in L_1$ and $x \leq y$ in L_1,

2. $x, y \in L_2$ and $x \leq y$ in L_2, or

3. $x \in L_1$ and $y \in L_2$.

Figure 13. The ordinal sum of two graded lattices.

It is straightforward to show that the ordinal sum of graded lattices is again a graded lattice, of rank $k+l+1$, and a maximal chain $C = x_0 < x_1 < \cdots < x_k < x_{k+1} < \cdots < x_{k+l+1}$ in $L_1 \oplus L_2$ is the concatenation of maximal chains $C_1 = x_0 < x_1 < \cdots < x_k$ in L_1 and $C_2 = x_{k+1} < x_{k+2} < \cdots < x_{k+l+1}$ in L_2. A pair of maximal chains in $L_1 \oplus L_2$, C and C', differ in precisely one element if and only if either

1. $C_1 = C_1'$ in L_1, and C_2 and C_2' differ in precisely one element in L_2, or

2. $C_2 = C_2'$ in L_2, and C_1 and C_1' differ in precisely one element in L_1.

When we consider the graphs associated to the order complexes of L_1 and L_2, the requirements described above are reminiscent of the definition of adjacency in the box product of two graphs seen in Chapter 2, and in fact we find this to be the case:

$$\Gamma_{max}^{k+l-2}(\Delta(L_1 \oplus L_2)) \simeq \Gamma_{max}^{k-3}(\Delta(L_1)) \Box \Gamma_{max}^{l-3}(\Delta(L_2)).$$

As we saw in the direct sum of lattices, the correspondence between maximal chains, and thus the isomorphism of graphs, is preserved when we consider lattices and their truncations, so we also have

$$\Gamma_{max}^{k+l-2}(\Delta(\overline{L_1 \oplus L_2})) \simeq \Gamma_{max}^{k-3}(\Delta(\overline{L_1})) \Box \Gamma_{max}^{l-3}(\Delta(\overline{L_2})).$$

We can therefore combine this relationship with Theorem 3.1 to obtain the following theorem.

Theorem 4.4. *Let L_1 and L_2 be two finite graded lattices of rank k and l, respectively. Let C be the concatenation of maximal chains C_1 and C_2 from L_1 and L_2, respectively. Then* $A_1^{k+l-2}(\Delta(\overline{L_1 \oplus L_2}), \overline{C}) \simeq A_1^{k-3}(\Delta(\overline{L_1}), \overline{C_1}) \times A_1^{l-3}(\Delta(\overline{L_2}), \overline{C_2})$.

4.3. The Ordinal Product of Two Lattices

The *ordinal product of* L_1 *and* L_2, $L_1 \otimes L_2$, is determined by the partial ordering on $\{(x, y) : x \in L_1, y \text{ and } \in L_2\}$ where $(x, y) \leq (x', y')$ if (i) $x = x'$ and $y \leq y'$, or (ii) $x < x'$. As described in [14], we can construct the Hasse diagram of $L_1 \otimes L_2$, by replacing each element x of L_1 with a copy $L_2(x)$ of L_2, and then connecting the maximal element in $L_2(x)$ to the minimal element in $L_2(y)$ if y covers x in L_1. The result is a graded lattice of rank $(k+1)(l+1) - 1 = kl + k + l$. We note that in general, $L_1 \otimes L_2 \not\simeq L_2 \otimes L_1$. If a maximal chain in L_1 has length k, then a maximal chain in $L_1 \otimes L_2$ consists of the k edges $e_1, e_2, \ldots, e_k$ of a maximal chain $C_1 \in L_1$, alternating with $k+1$ (possibly different) maximal chains, $C_2^1, C_2^2, C_2^3, \cdots, C_2^{k+1}$, from L_2:

$$C = C_2^1 \; e_1 \; C_2^2 \; e_2 \; C_2^3 \cdots e_k \; C_2^{k+1}.$$

Figure 14. The ordinal product of two graded lattices.

We associate each maximal chain in $L_1 \otimes L_2$, and each vertex in $\Gamma_{max}^{kl+k+l-3}(\Delta(\overline{L_1 \otimes L_2}))$ with the $(k+2)$-tuple $(C_1, C_2^1, C_2^2, C_2^3, \cdots, C_2^{k+1})$. Two maximal chains in $L_1 \otimes L_2$ correspond to adjacent vertices in $\Gamma_{max}^{kl+k+l-3}(\Delta(\overline{L_1 \otimes L_2}))$ if and only if they differ in precisely one element. If we assume that L_2 is not the lattice on a single element, then two maximal chains in $L_1 \otimes L_2$, C and C', differ in one element only if they differ in precisely one of the chains from L_2. If two $(k+2)$-tuples differ in the first coordinate, then the corresponding maximal chains are not adjacent in $L_1 \otimes L_2$. Thus if $C \sim C'$, then $C_1 = C_1'$ and one pair of chains, C_2^i and $C_2^{i}{'}$ for some i, $1 \leq i \leq k+1$, differ in one element. If we fix a single chain C_1 as the first coordinate of C and C', then k of the remaining coordinates are the same in both chains from $L_1 \otimes L_2$, and in the one coordinate where they differ, the entries are adjacent chains in L_2.

For example, suppose $C_1 = C_1'$, and $C_2^i = C_2^{i}{'}$ for $1 \leq i \leq k$. Then C_2^{k+1} and $C_2^{k+1}{'}$ must be maximal chains in L_2 that differ in precisely one element. When we consider the conditions required of the last $k+1$ coordinates of the $k+2$-tuples, we see that this is precisely the definition of the box product of $k+1$ copies of $\Gamma_{max}^{l-1}(\Delta(L_2))$, which we denote as $\Gamma_{max}^{l-1}(\Delta(L_2))^{k+1}$. That is, $C_2^i \sim C_2^{i}{'}$ for precisely one pair of chains from L_2, and $C_2^j = C_2^{j}{'}$ for the remaining k pairs. Assume $l > 0$ and let n be the number of maximal chains in L_1, then $\Gamma_{max}^{kl+k+l-3}(\Delta(\overline{L_1 \otimes L_2}))$ is isomorphic to n disjoint copies of $\Gamma_{max}^{l-1}(\Delta(L_2))^{k+1}$, and all vertices in a single copy have the same chain from L_1 for the first coordinate. Furthermore, since $\Gamma_{max}^{l-1}(\Delta(L_2)) \simeq \Gamma_{max}^{l-3}(\Delta(\overline{L_2}))$, $\Gamma_{max}^{kl+k+l-3}(\Delta(\overline{L_1 \otimes L_2}))$ is isomorphic to n disjoint copies of $\Gamma_{max}^{l-3}(\Delta(\overline{L_2}))^{k+1}$. Again, we use Theorem 3.1 to obtain the following result.

Theorem 4.5. *Let L_1 and L_2 be two finite graded lattices of rank k and l, respectively, with $l > 0$. Let $C = (C_1, C_2^1, C_2^2, C_2^3, \cdots, C_2^{k+1})$ be a maximal chain in $L_1 \otimes L_2$ as defined* *above. Then*

$$A_1^{kl+k+l-3}(\Delta(\overline{L_1 \otimes L_2}), \overline{C}) \simeq A_1^{l-3}(\Delta(\overline{L_2}), \overline{C_2^1}) \times A_1^{l-3}(\Delta(\overline{L_2}), \overline{C_2^2}) \times \cdots \times A_1^{l-3}(\Delta(\overline{L_2}), \overline{C_2^{k+1}}).$$

We note that the first coordinate of C determines which copy of the box product the base vertex of $A_1^{kl+k+l-3}(\Delta(\overline{L_1 \otimes L_2}), \overline{C})$ is in, and the other $k+1$ coordinates determine the base vertices for the A_1 groups in the direct product of groups on the right side of the isomorphism in the theorem.

4.4. The Direct Product of Two Lattices

Finally, we consider $L_1 \times L_2$, the direct product of L_1 and L_2, in order to use the isomorphism $B_n \simeq \mathbf{2}^n$, to help us understand the structure of $\Gamma_{max}^{n-3}(\Delta(\overline{B_n}))$. The elements of $L_1 \times L_2$ are the ordered pairs (x, y) where $x \in L_1$ and $y \in L_2$, with the partial order given by $(x, y) \leq (x', y')$ in $L_1 \times L_2$ if $x \leq x'$ in L_1 and $y \leq y'$ in L_2. As Stanley [14] points out, one way to construct the Hasse diagram of $L_1 \times L_2$ is to replace each element x of L_1 with a copy $L_2(x)$ of L_2 and connect corresponding elements in $L_2(x)$ and $L_2(x')$ if x and x' are connected in the diagram for L_1. If we view the Hasse diagram as a graph, one easily sees that this is equivalent to constructing the box product of the diagrams for L_1 and L_2, and that the result is a graded lattice of rank $k + l$.

Figure 15 shows an example of the direct product of two lattices, and it also illustrates the complexity of $\Gamma_{max}^{k+l-3}(\Delta(\overline{L_1 \times L_2}))$ in comparison to $\Gamma_{max}^{k-3}(\Delta(\overline{L_1}))$ and $\Gamma_{max}^{l-3}(\Delta(\overline{L_2}))$. There are two maximal chains of length 1 in $\overline{L_1}$, so $\Gamma_{max}^{0}(\Delta(\overline{L_1}))$ consists of a single edge, while $\overline{L_2}$ has a unique chain of length 1, and the corresponding graph $\Gamma_{max}^{0}(\Delta(\overline{L_2}))$ is a single vertex. Thus the box product $\Gamma_{max}^{0}(\Delta(\overline{L_1}))\Box\Gamma_{max}^{0}(\Delta(\overline{L_2}))$ is simply a single edge. On the other hand, $\overline{L_1 \times L_2}$ clearly has more than two chains of length 4, in fact, it has

Figure 15. The direct product of two lattices.

20 maximal chains. With 20 vertices, the graph $\Gamma^3_{max}(\Delta(\overline{L_1 \times L_2}))$ cannot be isomorphic to $\Gamma^0_{max}(\Delta(\overline{L_1}))\square\Gamma^0_{max}(\Delta(\overline{L_2}))$, and even for this small example it is clear that it is a much more complex graph.

To resolve this difficulty in constructing $\Gamma^{k+l-3}_{max}(\Delta(\overline{L_1 \times L_2}))$ for arbitrary lattices L_1 and L_2, we consider maximal chains in L_1, L_2, and $L_1 \times L_2$ before the minimal and maximal elements of each lattice are removed, and show that each maximal chain C in $L_1 \times L_2$ is a shuffle of a pair of maximal chains, $C_1 \in L_1$ and $C_2 \in L_2$. Let $u = u_1u_2u_3\cdots u_m \in S_{[m]}$ and $v = v_1v_2v_3\cdots v_n \in S_{[m+1,m+n]}$. A *shuffle* of u and v is a permutation $w_1w_2w_3\cdots w_{m+n}$ of $[m+n]$ such that $u_1u_2u_3\cdots u_m$ and $v_1v_2v_3\cdots v_l$ are subsequences of $w_1w_2w_3\cdots w_{m+n}$ (see [15]). There are $\binom{m+n}{m}$ possible shuffles of u and v. When we shuffle two chains, we shuffle the edges of the chains rather than the elements. Each edge in a maximal chain is associated with a pair of elements where one of the elements covers the other. To avoid adding more notation, we identify each edge by the associated element of greater rank.

For example, if we consider $C_1 = x_0 < x_1 < x_2 < x_4$ in L_1 and $C_2 = y_0 < y_1 < y_2 < y_3$ in L_2 from Figure 15, then we refer to the three edges in C_1 as x_1, x_2, and x_4, and the three edges in C_2 as y_1, y_2, and y_3. One possible shuffle of the edges of C_1 and C_2 is $x_1y_1x_2x_4y_2y_3$. We can associate this shuffle of C_1 and C_2 with a maximal chain C in $L_1 \times L_2$ as follows: We start with the minimal element (x_0, y_0) in $L_1 \times L_2$, and then make a sequence of 6 changes to the coordinates of the element, where the order of the changes is determined by the shuffle. First, we change the first coordinate to x_1, resulting in (x_1, y_0), which is the second element in C. Next, change the second coordinate

Figure 16. One shuffle of C_1 and C_2, associated with 5-sequence $\{1, 1, 3, 3, 3\}$ and 4-sequence $\{0, 2, 2, 5\}$.

to y_1, resulting in (x_1, y_1). The resulting sequence of elements after the six changes is $C = (x_0, y_0) < (x_1, y_0) < (x_1, y_1) < (x_2, y_1) < (x_4, y_1) < (x_4, y_2) < (x_4, y_3)$, which is a maximal chain in $L_1 \times L_2$.

When combining chains C_1 and C_2 from lattices L_1 and L_2 of rank k and l, respectively, we shuffle the k edges of C_1 with the l edges of C_2 to determine the order of the changes in the coordinates of the resulting chain C in $L_1 \times L_2$, thus there are $\binom{k+l}{k}$ possible combinations of the two chains. We refer to the resulting chain C as a shuffle of C_1 and C_2. We can associate a k-sequence and an l-sequence with a fixed shuffle, C, of C_1 and C_2 as follows: Color the k edges from C_1 red; these are the edges where the first coordinate of the element in C changes. Color the l edges from C_2 blue, where the second coordinate changes. Next, label each red edge with the number of blue edges below it in the chain. The ordered, weakly increasing collection of labels is the k-sequence associated with that shuffle. Similarly, label each blue edge with the number of red edges below it in the chain and the ordered collection of labels is an l-sequence. Clearly, the shuffle can be uniquely reconstructed using either the k-sequence or the l-sequence, but the information from both sets is be useful in the next section when we construct $\Gamma^{k+l-3}_{max}(\Delta(\overline{L_1 \times L_2}))$.

If we consider the example in Figure 16, the 5-sequence $\{1, 1, 3, 3, 3\}$ indicates that in the shuffle we "inserted" two edges from C_1 into C_2 at rank 1 and three edges at rank 3. Similarly the 4-sequence $\{0, 2, 2, 4\}$ indicates that 1 edge from C_2 was inserted into C_1 at rank 0, two at rank 2, and one at rank 4.

Figure 17. $\Gamma^{3,2}_{shuffle}$ labelled with 3-sequences and 2-sequences.

4.5. Constructing $\Gamma^{k+l-3}_{max}(\Delta(\overline{L_1 \times L_2}))$

In the process of computing the discrete fundamental group of $L_1 \times L_2$, the next step is to construct the associated graph $\Gamma^{k+l-3}_{max}(\Delta(\overline{L_1 \times L_2}))$. In order to do this, we first define three graphs: Γ_{L_1}, Γ_{L_2}, and $\Gamma^{k,l}_{shuffle}$. Next, we show that the vertices of the box product of these three graphs, $\widetilde{\Gamma}_{L_1 \times L_2} = \Gamma_{L_1} \Box \Gamma_{L_2} \Box \Gamma^{k,l}_{shuffle}$, correspond to maximal chains in $L_1 \times L_2$, but $\widetilde{\Gamma}_{L_1 \times L_2}$ has too many edges. Finally, we define a particular set of edges to remove from $\widetilde{\Gamma}_{L_1 \times L_2}$ and show that the resulting graph is the desired $\Gamma^{k+l-3}_{max}(\Delta(\overline{L_1 \times L_2}))$.

Step 1 Define Γ_{L_1}, Γ_{L_2}, and $\Gamma^{k,l}_{shuffle}$.

The vertices of Γ_{L_i} for $i \in \{1, 2\}$ correspond to maximal chains in L_i, and two vertices are adjacent if and only if the two corresponding chains are adjacent in L_i. Recall that for a lattice L of rank k, $\Gamma^{k-1}_{max}(\Delta(L)) \simeq \Gamma^{k-3}_{max}(\Delta(\overline{L}))$. Lattices L_1 and L_2 have rank k and l, respectively, and for a pair of chains, C_1 from L_1 and C_2 from L_2, there are $\binom{k+l}{k}$ ways to shuffle C_1 with C_2 to get a maximal chain in $L_1 \times L_2$. The vertices of the *shuffle graph for k and l*, $\Gamma^{k,l}_{shuffle}$, correspond to the $\binom{k+l}{k}$ shuffles. Label each vertex with the pair (k-sequence, l-sequence) that corresponds to each shuffle.

Two vertices in $\Gamma^{k,l}_{shuffle}$ are adjacent if and only if we can reverse the order of a pair of consecutive edges in the shuffle corresponding to one vertex, one edge from C_1 and the other from C_2, to obtain the chain corresponding to the other shuffle. In this case, we say that the two corresponding k-sequences *differ by a single change*, that is, one pair of corresponding elements in the sequences differs by 1, and all other elements in the two k-

sequences are the same. We say that two shuffles are *adjacent* if they correspond to adjacent vertices in $\Gamma^{k,l}_{shuffle}$. For example, the two 5-sequences $\{1,1,3,3,3\}$ and $\{1,2,3,3,3\}$ differ by a single change and correspond to adjacent shuffles. Note that if two k-sequences differ by a single change, then the associated l-sequences also differ by a single change, so we only need to refer to one of the sequences when determining if two shuffles are adjacent.

Step 2 $\widetilde{\Gamma}_{L_1 \times L_2} = \Gamma_{L_1} \square \Gamma_{L_2} \square \Gamma^{k,l}_{shuffle}$.

We now define $\widetilde{\Gamma}_{L_1 \times L_2} = \Gamma_{L_1} \square \Gamma_{L_2} \square \Gamma^{k,l}_{shuffle}$ as an intermediate graph in our process of constructing $\Gamma^{k+l-3}_{max}(\Delta(\overline{L_1 \times L_2}))$. Label each vertex of $\widetilde{\Gamma}_{L_1 \times L_2}$ with the ordered triple $(C_1, C_2, (k\text{-sequence}, l\text{-sequence}))$. The set of vertices of $\widetilde{\Gamma}_{L_1 \times L_2}$ corresponds to all possible shuffles of pairs of maximal chains from L_1 and L_2, thus there is a one-to-one correspondence between the vertices of $\widetilde{\Gamma}_{L_1 \times L_2}$ and the maximal chains of $L_1 \times L_2$. From the definition of a box product of graphs, two vertices in $\widetilde{\Gamma}_{L_1 \times L_2}$, $(C_1, C_2, (k\text{-sequence}, l\text{-sequence}))$ and $(C_1', C_2', (k\text{-sequence}', l\text{-sequence}'))$, are adjacent if they satisfy precisely one of the following conditions:

1. $C_1 = C_1'$, $C_2 = C_2'$, and the two k-sequences differ by a single change.

2. $C_1 = C_1'$, $C_2 \sim C_2'$ in L_2, and k-sequence $=$ k-sequence$'$.

3. $C_1 \sim C_1'$ in L_1, $C_2 = C_2'$, and k-sequence $=$ k-sequence$'$.

Let $|\Gamma|$ denote the number of vertices in Γ, and let $\|\Gamma\|$ denote the number of edges in the graph. Each vertex in $\widetilde{\Gamma}_{L_1\times L_2}$ corresponds to a vertex from each of the three graphs in the box product and each edge in $\widetilde{\Gamma}_{L_1\times L_2}$ corresponds to a vertex from each of two of the three graphs and an edge from the remaining graph, therefore $|\widetilde{\Gamma}_{L_1\times L_2}| = |\Gamma_{L_1}| \cdot |\Gamma_{L_2}| \cdot \binom{k+l}{k}$ and $\|\widetilde{\Gamma}_{L_1\times L_2}\| = |\Gamma_{L_1}| \cdot |\Gamma_{L_2}| \cdot \|\Gamma^{k,l}_{shuffle}\| + |\Gamma_{L_1}| \cdot \|\Gamma_{L_2}\| \cdot \binom{k+l}{k} + \|\Gamma_{L_1}\| \cdot |\Gamma_{L_2}| \cdot \binom{k+l}{k}$.

Step 3 Removing edges from $\widetilde{\Gamma}_{L_1\times L_2}$.

Each edge in $\widetilde{\Gamma}_{L_1\times L_2}$ can be classified as type 1, 2, or 3, according to which of the above conditions is satisfied by $(C_1, C_2, (k\text{-sequence}, l\text{-sequence}))$ and $(C_1', C_2', (k\text{-sequence}', l\text{-sequence}'))$. We examine each type of edge to determine which ones correspond to edges between a pair of adjacent chains in $L_1 \times L_2$ and which do not. Edges corresponding to a pair of non-adjacent chains must be removed from the graph.

Type 1 edges. $C_1 = C_1'$, $C_2 = C_2'$, and the two k-sequences differ by a single change. The two vertices incident to an edge of type 1 correspond to maximal chains C and C' in $L_1 \times L_2$ constructed from the same pair, C_1 and C_2, of maximal chains from L_1 and L_2, where the chains are combined using adjacent shuffles. This is equivalent to reversing the order of a pair of consecutive edges in a chain in $L_1 \times L_2$, where one edge is from C_1 and the other is from C_2. Reversing the order of the two edges forms a diamond in the diagram of C and C' in $L_1 \times L_2$. Figure 18 shows chains C_1 and C_2 from Figure 16 combined in this way with adjacent shuffles associated to 5-sequences $\{1, 1, 3, 3, 3\}$ and $\{1, 1, 2, 3, 3\}$. The red edges are from C_1, and the blue edges are from C_2, and we see a diamond in the graph

Figure 18. Chains C_1 with C_2 combined using adjacent shuffles associated to 5-sequences $\{1, 1, 3, 3, 3\}$ and $\{1, 1, 2, 3, 3\}$.

Figure 19. Combining C_1 with C_2 and C_2' using two different shuffles.

where the order of a pair of consecutive edges, one red and one blue, has been reversed. These two resulting chains are adjacent in $L_1 \times L_2$, so we do not remove any edges of type 1 from $\widetilde{\Gamma}_{L_1 \times L_2}$.

Type 2 edges. $C_1 = C_1'$, $C_2 \sim C_2'$ in L_2, and k-sequence $=$ k-sequence$'$. The diagram of C_2 and C_2' in L_2 contains a diamond at rank i where the two chains differ. When we shuffle C_2 and C_2' with C_1, this diamond may be "stretched" by the insertion of edges from C_1, depending on which shuffle is used. Consider the k-sequence corresponding to the shuffle used. If the k-sequence does not contain an element i, then when C_2 and C_2' are shuffled with C_1, no edges from C_1 are inserted into C_2 and C_2' inside the diamond. The resulting chains are adjacent in $L_1 \times L_2$, so this edge is not removed from $\widetilde{\Gamma}_{L_1 \times L_2}$. However, if the k-sequence contains one or more elements i, then the shuffle inserts one or more edges into C_2 and C_2' at rank i, stretching the diamond. The resulting chains in $L_1 \times L_2$ differ by at least two elements, thus they are not adjacent and so this edge must be removed from $\widetilde{\Gamma}_{L_1 \times L_2}$. Figure 19 shows the result of combining C_1 with both C_2 and C_2' using the shuffles associated with 5-sequences $\{1, 1, 3, 3, 3, \}$ and $\{1, 2, 2, 3, 3\}$. The result of the first shuffle is a pair of chains that are adjacent in $L_1 \times L_2$, but the chains resulting from the second shuffle are not adjacent.

Type 3 edges. $C_1 \sim C_1'$ in L_1, $C_2 = C_2'$, and k-sequence $=$ k-sequence$'$. As in the analysis of type 2 edges, we must first identify the rank i where C_1 and C_1' differ, and then the l-sequence is used to determine if the shuffle of C_1 and C_1' with C_2 results in adjacent chains in $L_1 \times L_2$. If the l-sequence does not contain an element i, then the resulting chains are adjacent and the edge in $\widetilde{\Gamma}_{L_1 \times L_2}$ is retained. If the l-sequence contains one or more elements i, then the chains are not adjacent in $L_1 \times L_2$ and the edge is removed from $\widetilde{\Gamma}_{L_1 \times L_2}$. This completes our determination of which edges to remove from $\widetilde{\Gamma}_{L_1 \times L_2}$.

To calculate the total number of edges removed from $\widetilde{\Gamma}_{L_1 \times L_2}$, first consider an edge of type 2 and its incident vertices. There are $|\Gamma_{L_1}|$ choices for the first coordinate, C_1, in the pair of vertices, and $\|\Gamma_{L_2}\|$ choices for the adjacent pair C_2 and C_2' for the second coordinate. We can then choose a $(k-1)$-combination of $\{0, 1, 2, ..., l\}$ with repetition, and add the element i to it, guaranteeing that the resulting k-set has at least one element i. Thus there are $\left(\binom{l+1}{k-1}\right) = \binom{k+l}{k-1}$ choices of k-sets for the third coordinate such that the two vertices correspond to non-adjacent chains in $L_1 \times L_2$. Consequently a total of $|\Gamma_{L_1}| \cdot \|\Gamma_{L_2}\| \cdot \binom{k+l}{k-1}$ edges of type 2 are removed from $\widetilde{\Gamma}_{L_1 \times L_2}$. By a similar argument, $\|\Gamma_{L_1}\| \cdot |\Gamma_{L_2}| \cdot \binom{k+l}{l-1}$ edges of type 3 are also removed.

Let Γ be the resulting graph after any edges in $\widetilde{\Gamma}_{L_1 \times L_2}$ corresponding to non-adjacent pairs of maximal chains in $L_1 \times L_2$ have been removed. We now show that this is, in fact, $\Gamma_{max}^{k+l-3}(\Delta(\overline{L_1 \times L_2}))$. Before we began removing edges from $\widetilde{\Gamma}_{L_1 \times L_2}$, we noted that the vertices of $\widetilde{\Gamma}_{L_1 \times L_2}$, and therefore of Γ, correspond to the maximal chains in $L_1 \times L_2$. Except in the case where one lattice has a single element and the other has one or two elements, and thus $L_1 \times L_2$ has only one or two elements, there is a one-to-one correspondence between maximal chains in $L_1 \times L_2$ and maximal chains in $\overline{L_1 \times L_2}$: a maximal chain C in $L_1 \times L_2$ corresponds to $\overline{C} = C - \{(\hat{0}_1, \hat{0}_2), (\hat{1}_1, \hat{1}_2)\}$ in $\overline{L_1 \times L_2}$. Therefore the vertices of Γ also correspond to maximal chains in $\overline{L_1 \times L_2}$, furthermore, adjacency is preserved by this

Figure 20. Diamonds illustrating the three possible types of pairs of adjacent chains in $L_1 \times L_2$.

correspondence.

To show that Γ is the desired graph $\Gamma_{max}^{k+l-3}(\Delta(\overline{L_1 \times L_2}))$, we must also show that two vertices in Γ are adjacent if and only if the two corresponding chains in $\overline{L_1 \times L_2}$ are adjacent. We have shown that each edge in $\widetilde{\Gamma}_{L_1 \times L_2}$ that we kept corresponds to a pair, C and C', of adjacent maximal chains in $L_1 \times L_2$, which in turn corresponds to a pair of adjacent maximal chains $\overline{C}$ and $\overline{C'}$ in $\overline{L_1 \times L_2}$. Now suppose $\overline{C}$ and $\overline{C'}$ are a pair of adjacent maximal chains in $\overline{L_1 \times L_2}$; it is sufficient to show that the corresponding adjacent chains C and C' in $L_1 \times L_2$ correspond to an edge in Γ. Chains $\overline{C}$ and $\overline{C'}$, and consequently C and C', differ in precisely one element; suppose they differ at rank i for some $1 \leq i \leq k+l-1$; note that they have the same elements at rank $i-1$ and $i+1$. This can occur in one of the following three ways.

Case 1. From rank $i-1$ to rank $i+1$, there is one change made in the first coordinate of each chain, and one change made to the second coordinate, but the changes are made in the opposite order in the two chains. Without loss of generality, assume that in C, the first coordinate changes from rank $i-1$ to rank i, and the second coordinate changes from rank i to $i+1$; in C', the second coordinate changes from rank $i-1$ to i, and the first coordinate changes from rank i to $i+1$. If we look at just these three ranks in C and C' we get a diamond as shown in case 1 in Figure 20. This means that when we shuffled C_1 with C_2 we used adjacent shuffles to form C and C'. This corresponds to a type 1 edge in $\widetilde{\Gamma}_{L_1 \times L_2}$, none of which were removed when constructing Γ.

<u>Case 2.</u> The only changes made were to the second coordinates at ranks i and $i+1$. Element $y_{n+1} \neq y'_{n+1}$, but they both cover y_n and both are covered by y_{n+2} in L_2. Considering ranks $i-1$ to $i+1$, we get the second diamond shown in Figure 20. The changes in the first and second coordinates were made in the same order in C and C', therefore they result in part from two adjacent chains C_2 and C'_2. These chains were combined with a single chain C_1 from L_1 and the same shuffle, otherwise the resulting C and C' would differ in other elements as well. Furthermore, the shuffle used to combine C_1 with C_2 and C'_2 did not insert any edges from C_1 into C_2 and C'_2 at rank j, where C_2 and C'_2 differ in L_2, so the corresponding k-sequence that records these ranks does not contain any elements j. This situation corresponds to an edge of type 2 that was not removed from $\widetilde{\Gamma}_{L_1 \times L_2}$.

<u>Case 3.</u> This is analogous to case 2, but the only changes from rank $i-1$ to rank $i+1$ were made to the first coordinate of elements in C and C'. Similarly, $x_{m+1} \neq x'_{m+1}$, but both cover x_m and are covered by x_{m+2} in L_1. Then C and C' were constructed by combining adjacent chains C_1 and C'_1 with a single chain C_2 using a shuffle whose corresponding l-sequence doesn't contain any elements corresponding to the rank where C_1 and C'_1 differ in L_1. This corresponds to an edge of type 3 that was not removed from Γ'. These are the only ways in which two adjacent chains C and C' in $L_1 \times L_2$ can differ, so if $C \sim C'$, then there is an edge between the corresponding vertices in Γ. Therefore, the graph resulting from the construction described in this section in precisely $\Gamma^{k+l-3}_{max}(\Delta(\overline{L_1 \times L_2}))$.

CHAPTER 5

THE BOOLEAN LATTICE

5.1. Building Γ_{B_n} from Smaller Graphs

While it is not immediately clear from the construction of $\Gamma_{max}^{k+l-3}(\Delta(\overline{L_1 \times L_2}))$ if there is an easily defined relationship between the groups $A_1^{k-3}(\Delta(\overline{L_1}))$, $A_1^{l-3}(\Delta(\overline{L_2}))$, and $A_1^{k+l-3}(\Delta(\overline{L_1 \times L_2}))$, the construction defined in the previous chapter proves to be instrumental in understanding the structure of the discrete fundamental group for the Boolean lattice. The elements of B_n, the Boolean lattice of rank n for $n \geq 1$, are the 2^n subsets of $[n]$, ordered by inclusion. A maximal chain C in B_n is a sequence of elements $x_0 < x_1 < x_2 < \cdots < x_n$ where each element x_i in the chain is a subset of $[n]$ of size i. The lattice B_n has $n!$ maximal chains, which can be put into one-to-one correspondence with the permutations in S_n, the symmetric group on n elements, by defining $\sigma_C(i) = x_i - x_{i-1}$ for $i \in [n]$. In general, when referring to permutations, we use single line notation unless otherwise indicated; for example, the chain $C = \phi, \{2\}, \{2,4\}, \{1,2,4\}, \{1,2,3,4\}$ in B_4 is identified with the permutation 2413 in S_4.

Building $\widetilde{\mathbf{\Gamma}}_{\mathbf{B_n}} = \mathbf{\Gamma}_{\mathbf{B_{n-1}}} \square \mathbf{\Gamma_2} \square \mathbf{\Gamma}^{\mathbf{n-1,1}}_{\mathbf{shuffle}}$. We now use the construction of $\Gamma_{max}^{k+l-3}(\Delta(\overline{L_1 \times L_2}))$ described in the previous chapter to give us a better understanding of $\Gamma_{max}^{n-3}(\Delta(\overline{B_n}))$ and make it possible to compute $A_1^{n-3}(\Delta(\overline{B_n}))^{ab}$. For ease of notation, we simply refer to $\Gamma^{n-3}_{max}(\Delta(\overline{B_n}))$ as Γ_{B_n}. Recall that $B_n \simeq \mathbf{2}^n$ where $\mathbf{2}$ is the poset on two elements, x and y, with $x < y$. It is useful to express this isomorphism as $B_n \simeq B_{n-1} \times \mathbf{2}$, because when we construct $\widetilde{\Gamma}_{B_n} = \Gamma_{B_{n-1}} \square \Gamma_{\mathbf{2}} \square \Gamma^{n-1,1}_{shuffle}$ the graph $\Gamma_{\mathbf{2}}$ consists of a single vertex corresponding to C, where C denotes the unique maximal chain of length 1 in $\mathbf{2}$. This does not increase the total number of vertices in $\Gamma_{B_{n-1}} \square \Gamma_{\mathbf{2}} \square \Gamma^{n-1,1}_{shuffle}$ in comparison to $\Gamma_{B_{n-1}} \square \Gamma^{n-1,1}_{shuffle}$, and the ordered triple for every vertex in $\widetilde{\Gamma}_{B_n}$ has C as its second coordinate. Additionally, the graph $\Gamma^{n-1,1}_{shuffle}$ is a single path of length $n-1$; when we shuffle a maximal chain C_1 of length $n-1$ from B_{n-1} with chain C from $\mathbf{2}$ to form a chain of length n in $B_{n-1} \times \mathbf{2}$, we can insert the single edge from C into C_1 anywhere from rank 0 to rank $n-1$. The 1-sequence associated to each of the shuffles indicates at which rank this occurs, and two shuffles are adjacent if the insertion of the edge from C occurs at consecutive ranks. Thus we see that $\widetilde{\Gamma}_{B_n}$ is isomorphic to the box product of $\Gamma_{B_{n-1}}$ and a path of length $n-1$.

When constructing and describing $\widetilde{\Gamma}_{B_n}$, we use the language of permutations rather than chains, so we need to make some adjustments to the labeling of the vertices of $\widetilde{\Gamma}_{B_n}$. Each vertex is initially labeled with the triple $(C_1, C, ((n-1)\text{-sequence}, 1\text{-sequence}))$. We replace C_1 with the corresponding permutation σ_{C_1}, or simply σ, in S_{n-1}, and we replace C with the single element n. Furthermore, we see that we only need to consider the 1-sequence, an element $i \in \{0, 1, 2, \ldots, n-1\}$, when determining which edges to remove from $\widetilde{\Gamma}_{B_n}$. Consequently, the new labeling for each vertex is simplified to (σ, n, i). We may now associate each vertex in $\widetilde{\Gamma}_{B_n}$ with a shuffle of σ with n, which is a permutation in S_n in single line notation. Figure 21 shows $\widetilde{\Gamma}_{B_4}$ with the vertices labeled using permutations in single line notation. The 1-sequence determines where we insert n into the permutation σ by indicating how many entries of σ come before n in the shuffle.

Figure 21. The intermediate graph $\widetilde{\Gamma}_{B_4}$.

We view $\widetilde{\Gamma}_{B_n}$ as having n *levels*, where each level is one of the n copies of $\Gamma_{B_{n-1}}$ used in constructing $\widetilde{\Gamma}_{B_n}$. Each level corresponds to a distinct shuffle, so we number the levels from 1 to n according to the position of the element n in all permutations at that level. Consecutive levels correspond to adjacent shuffles, so each vertex in level i is connected to the corresponding vertex in level $i+1$ by an edge for $1 \leq i \leq n-1$. We classify each edge in $\widetilde{\Gamma}_{B_n}$ as either *horizontal* if it is incident to two vertices within a single level of $\widetilde{\Gamma}_{B_n}$, or *vertical* if it is incident to one vertex in level i and the corresponding vertex in level $i+1$ for some $1 \leq i \leq n-1$.

Removing edges from $\widetilde{\boldsymbol{\Gamma}}_{\mathbf{B_n}}$. We must remove edges in $\widetilde{\Gamma}_{B_n}$ that do not correspond to adjacent permutations in S_n. We show that the edges we remove are precisely the horizontal edges in level i, for $2 \leq i \leq n-1$, where the corresponding permutations σ and σ' first differ in position $i-1$.

Type 1 edges. The edges of type 1 correspond to a pair of vertices where the associated shuffles are adjacent. Here we are using the 1-sequence corresponding to each shuffle, an element in $\{0, 1, 2, \ldots, n-1\}$, and two shuffles are adjacent if the 1-sequences are consecutive integers i and $i+1$ for some $0 \leq i \leq n-2$. The vertices incident to a horizontal edge are within a single level and are therefore associated to the same shuffle, whereas the vertices incident to a vertical edge are in consecutive levels and are associated to adjacent shuffles. Thus the edges of type 1 are precisely the set of vertical edges in $\widetilde{\Gamma}_{B_n}$. None of these vertical edges are removed from $\widetilde{\Gamma}_{B_n}$.

Type 2 edges. In general, type 2 edges correspond to a pair of vertices where the second coordinates are adjacent chains (permutations). However, in the case of B_n, all vertices in $\widetilde{\Gamma}_{B_n}$ are labeled (σ, n, i), so we have no edges of type 2 in $\widetilde{\Gamma}_{B_n}$.

Type 3 edges. Each of the remaining edges, of type 3, corresponds to a pair of vertices where $\sigma \sim \sigma'$. In particular, both vertices are associated with the same shuffle and thus are in the same level of $\widetilde{\Gamma}_{B_n}$. To determine which of these edges to remove, we identify the position, i, where σ and σ' first differ. If the 1-sequence is the same element i, then we remove that edge. There is no position 0 in a permutation, so we do not remove any edges associated to the shuffle identified by $i = 0$, which are the horizontal edges in level 1. Furthermore, two permutations cannot differ only in the last position, so we do not remove any edges at level n, either. Thus we remove the horizontal edges in level i, for $2 \leq i \leq n-1$, where the corresponding permutations σ and σ' first differ in position $i-1$, because when we insert n into σ and σ' the corresponding chains are not adjacent in B_n.

In the previous chapter, we observed that we removed $\|\Gamma_{L_1}\| \cdot |\Gamma_{L_2}| \cdot \binom{k+l}{l-1}$ edges of type 3 from $\widetilde{\Gamma}_{L_1 \times L_2}$. In the case of $\widetilde{\Gamma}_{B_n}$, the rank of B_{n-1} is $n-1$, and the rank of $\mathbf{2}$ is 1. In the next section, we will see that Γ_{B_n} has $n!$ vertices and is $(n-1)$-regular, so $\|\Gamma_{B_{n-1}}\| = \frac{(n-1)!(n-2)}{2}$. Thus the total number of edges removed from $\widetilde{\Gamma}_{B_n}$ is

$$\|\Gamma_{B_{n-1}}\| \cdot |\Gamma_{\mathbf{2}}| \cdot \binom{n-1+1}{1-1} = \frac{(n-1)!(n-2)}{2} \cdot 1 \cdot \binom{n}{0} = \frac{(n-1)!(n-2)}{2}.$$

5.2. The Structure of Γ_{B_n}

Now that we have removed all appropriate edges from $\widetilde{\Gamma}_{B_n}$, the resulting graph is $\Gamma^{n-3}_{max}(\Delta(\overline{B_n}))$, which we simply refer to as Γ_{B_n}. The vertices correspond to maximal chains in B_n and in $\overline{B_n}$ as well, or equivalently, permutations in S_n. The pair of permutations incident to an edge, σ_1 and σ_2, differ by a switch of elements in consecutive positions, thus Γ_{B_n} is the 1-skeleton of the permutahedron Π_{n-1} [17]. This difference corresponds to multiplication on the right by a simple transposition $(i, i+1)$ for some i, $1 \leq i \leq n-1$. For example, in Γ_{B_4}, $\sigma_1 = 2413$ is adjacent to $\sigma_2 = 2143$, (respectively (1243) and (12)(34) in cycle notation), and we note that (1243)=(12)(34)(23) and similarly (12)(34)=(1243)(23). We therefore say that σ_1 and σ_2 are *adjacent* permutations, denoted by $\sigma_1 \sim \sigma_2$, and that they *differ by the simple transposition* (23).

Furthermore, we associate each edge in Γ_{B_n} with a simple transposition in S_n. Each vertex in Γ_{B_n} is incident to precisely one edge for each of the $n-1$ simple transpositions in S_n, so Γ_{B_n} is $(n-1)$-regular. Figure 22 shows Γ_{B_4} with vertices labeled with the associated permutations and edges colored according to the associated transpositions. Note that this differs from the coloring of edges of chains that we used in Chapter 4. We see that in Γ_{B_4} (and in Γ_{B_n} in general), the vertical edges between levels i and $i+1$ are associated to the transposition $(i, i+1)$.

Figure 22. The final graph Γ_{B_4}.

If $\sigma_1 = \sigma_2(i, i+1)$ and σ_1 is odd, then σ_2 is even. Therefore, Γ_{B_n} is a bipartite graph with the vertices partitioned into even and odd permutations in S_n, and consequently all cycles in Γ_{B_n} are even, something well known. The graph Γ_{B_n} is simple, so it contains no cycles of length 2. A cycle of length 4 may be associated with a pair of *disjoint* simple transpositions, which are of the form $(i, i+1)$ and $(j, j+i)$ where $|i-j| \geq 2$. For example, the cycle 1234-2134-2143-1243 in Γ_{B_4} is associated with the transpositions (12) and (34). Disjoint transpositions commute, so the sequence of transpositions associated to the edges of this cycle corresponds to expressing the identity element in S_4 as $(12)(34)(12)(34)$. Two transpositions that share one element do not commute, so $(12)(23)(12)(23)(12)(23)$ is an irreducible representation of the identity element. This representation corresponds to a 6-cycle in Γ_{B_n}, such as 1234-2134-2314-3214-3124-1324, which we call a *reduced 6-cycle* because it corresponds to a reduced representation of the identity element and it is not the product of two or more 4-cycles. In general, when we refer to 6-cycles, we mean reduced 6-cycles.

It is known that the set of simple transpositions $(i, i+1)$ for $1 \leq i \leq n-1$ and the relations $(i, i+1)^2 = 1$, $((i, i+1)(i+1, i+2))^3 = 1$, and $(i, i+1)(j, j+1) = (j, j+1)(i, i+1)$ if $|i-j| \geq 2$, form a presentation for S_n [8]. Thus all other cycles in Γ_{B_n} of length ≥ 8, which also correspond to representations of the identity in S_n, can be expressed as the concatenation of cycles of length 4 or 6. A 4-cycle is G-homotopic to a single vertex, so we turn our attention to the equivalence classes of 6-cycles in Γ_{B_n} in order to compute the number of generators of $A_1^{n-3}(\Delta(\overline{B_n}))^{ab}$.

CHAPTER 6

EQUIVALENCE CLASSES OF 6-CYCLES IN Γ_{B_n}

In this chapter, we demonstrate how to distinguish between different G-homotopy equivalence classes so that we may count them. First, we show that two 6-cycles in the same equivalence class are associated with the same pair of adjacent transpositions $(i-1, i)$ and $(i, i+1)$ for some i, $2 \leq i \leq n-1$. We then prove a stronger theorem: two 6-cycles C_1 and C_2 are in the same equivalence class if and only if they differ by a sequence of transpositions disjoint from $(i-1, i)$ and $(i, i+1)$, that is, if we multiply all six permutations in C_1 by the same sequence of transpositions, the result is precisely the six permutations in C_2. This theorem, when combined with our new understanding of the structure of Γ_{B_n}, gives us the means to describe the equivalence classes, and to find a formula for the rank of $A_1^{n-3}(\Delta(\overline{B_n}))^{ab}$. Recall that this is also the first Betti number of $M_{n,3}$, which Björner and Lovász showed will give us a lower bound for the complexity of the k-equal problem.

6.1. Each Equivalence Class is Associated with a Pair of Transpositions

In the previous chapter, we noted that each (reduced) 6-cycle is associated to a pair of adjacent transpositions $(i-1, i)$ and $(i, i+1)$ for some i, $2 \leq i \leq n-1$. We also described edges in Γ_{B_n} as horizontal if they are incident to permutations in the same level, or vertical if they are incident to permutations in consecutive levels of the graph. We may similarly classify 6-cycles in Γ_{B_n} as *horizontal* if all six permutations in the cycle are in the same level of Γ_{B_n}. These horizontal 6-cycles originated in one of the n copies of $\Gamma_{B_{n-1}}$ used when we constructed $\widetilde{\Gamma}_{B_n}$. However, not all of these 6-cycles exist in Γ_{B_n}, because some of them were "broken" when we removed edges from $\widetilde{\Gamma}_{B_n}$ to create Γ_{B_n}. The graph Γ_{B_3} is a single 6-cycle, but the copies of that cycle at levels 2 and 3 of $\widetilde{\Gamma}_{B_4}$ were broken as a result of the removal of edges.

At the same time, removing edges created new 6-cycles, such as 1234-1324-1342-1432-1423-1243, that contain two permutations from each of three consecutive levels in Γ_{B_4}. We call a cycle of this type, containing two vertices from each of three levels, $i-1$, i, and $i+1$, and created by the removal of an edge at level i, a *vertical cycle at level* i. There are no 6-cycles that span four levels of Γ_{B_n}, because this would require at least one permutation to be adjacent to two permutations in a single neighboring level, which is not possible in Γ_{B_n}. Nor are there any 6-cycles comprised of permutations in precisely two levels of the graph. We can't have a 6-cycle with permutations in only two neighboring levels. No vertical edges were removed from $\widetilde{\Gamma}_{B_n}$, so we would have three vertical edges, and the 6-cycle would be the concatenation of two 4-cycles. Nor can we have a 6-cycle with two permutations in one level and four vertices in a neighboring level. Suppose the two vertical edges between pairs of permutations in the two levels are of type $(i-1, i)$. The cycle must contain a third edge of type $(i-1, i)$, which must be a horizontal edge in level i, but level i contains no horizontal edges of type $(i-1, i)$ or $(i, i+1)$. Therefore, all 6-cycles in Γ_{B_n} may be classified as either horizontal or vertical.

Suppose we have two reduced 6-cycles, C_1 and C_2. How can we tell if they are G-homotopic to one another? We know that each cycle is associated with a pair of simple

Figure 23. Vertical 6-cycles in Γ_{B_n}.

transpositions of the form $(i-1,i)$ and $(i,i+1)$ for some i, $2 \leq i \leq n-1$, but can C_1 and C_2 be associated to different pairs of transpositions, or must they be associated to the same pair? To answer these questions, we assume that we have a G-homotopy from C_1 to C_2 and we consider the acceptable types of changes that we can make from row to row in a G-homotopy grid, as well as the impact of these changes on the images of consecutive rows of the grid. We find that all of these acceptable changes preserve the parity of the number of edges in each row that are associated to a given transposition. For example, if the first row of the grid contains an odd number of edges associated with the transposition (12), then every row contains an odd number of edges associated with (12). This leads to the conclusion that if $C_1 \simeq_G C_2$, then C_1 and C_2 must in fact be associated to the same pair of simple transpositions. A further consequence of this lemma is that a reduced 6-cycle cannot be contracted to a single vertex. A valid G-homotopy grid cannot contain an odd number of edges associated with each of $(i-1,i)$ and $(i,i+1)$ in the first row, and no edges associated with either transposition in the last row, or any row, for that matter.

Lemma 6.1. *Let C_1 and C_2 be two distinct reduced 6-cycles in Γ_{B_n}. If $C_1 \simeq_G C_2$, then they are associated with the same pair of transpositions, $(i-1,i)$ and $(i,i+1)$, for some i, $2 \leq i \leq n-1$.*

Proof. Let $C_1 = \sigma_0 - \sigma_1 - \sigma_2 - \sigma_3 - \sigma_4 - \sigma_5 - \sigma_0$ and $C_2 = \gamma_0 - \gamma_1 - \gamma_2 - \gamma_3 - \gamma_4 - \gamma_5 - \gamma_0$ be two distinct reduced 6-cycles in Γ_{B_n} and suppose $C_1 \simeq_G C_2$. The graph Γ_{B_n} is connected, so $A_1^G(\Gamma_{B_n})$ is independent of the choice of the base vertex, thus we may choose σ_0 as the base vertex (base permutation). We must be able to construct a G-homotopy from C_1 to

C_2; the first row of the G-homotopy grid corresponds to C_1 and the last row corresponds to PC_2P^{-1}, where P is a path from σ_0 to γ_0 in Γ_{B_n}. Each vertex in the G-homotopy grid is labeled with its image, a permutation in S_n, and an edge in the grid may be incident to two copies of the same permutation or incident to adjacent permutations. In the latter case, we can label the edge with the associated transposition, and we leave the other edges unlabeled. (In the figures in this chapter, we label vertices and color edges to avoid confusion about what the labels refer to. Black edges are incident to two copies of the same permutation.)

Recall that the image of each row in the G-homotopy grid is a loop in Γ_{B_n} based at σ_0, our base permutation. Therefore the sequence of transpositions labeling the edges in a single row is a representation of the identity in S_n. In this proof we consider the changes that we can make to a sequence of transpositions so that it is still a representation of the identity, and so that the changes are consistent with the definition of a G-homotopy. The possible changes are limited to shifting labeled edges in the grid, inserting or deleting a pair of transpositions of the form $(j, j+1)(j, j+1)$, commuting disjoint transpositions, or a combination of these changes. We examine the results of each of the possible changes with respect to two consecutive rows in the grid: the effect on the sequence of labeled edges, and the images of the two rows in Γ_{B_n}.

Suppose that C_2 is associated with transpositions $(i-1, i)$ and $(i, i+1)$ for some i, $2 \leq i \leq n-1$. Without loss of generality, we assume that the first edge in C_2, incident to γ_0 and γ_1, is associated with $(i-1, i)$. The last row of the G-homotopy grid, the one corresponding to PC_2P^{-1}, has a series of transpositions labeling the edges of P, then the alternating subsequence $(i-1, i)(i, i+1)(i-1, i)(i, i+1)(i-1, i)(i, i+1)$ corresponding to the six edges in C_2, followed by the transpositions from P in reverse order. The path P may contain copies of transpositions $(i-1, i)$ or $(i, i+1)$, but we note that there are three copies of $(i-1,i)$ from C_2 and an even number of copies, if any, from P and P^{-1} combined. Therefore, the last row of the grid contains an odd number of transpositions $(i-1,i)$, and similarly, an odd number of transpositions $(i,i+1)$.

We show that all of the changes that we can make from row to row in a G-homotopy grid preserve the parity of number of edges in each row that are associated with each simple transposition. Consequently, each row of the grid, and in particular the first row, which corresponds to C_1, also contains an odd number of each of these two transpositions. We know that C_1 is a 6-cycle corresponding to a pair of adjacent transpositions, and therefore if the row corresponding to C_1, which contains precisely six labeled edges, contains at least one edge of each type, $(i-1,i)$ and $(i,i+1)$, then C_1 must be associated with that pair of transpositions.

Case 1 Recall that unlabeled edges in a G-homotopy grid are incident to two copies of the same vertex (permutation) in Γ_{B_n}. If there are unlabeled edges in a row of the grid, we may shift one or more labeled edges one position to the right or left in the following row. This does not change the image from one row to the next, or the representation of the identity, it simply changes where the repetition of one or more vertices occurs in the row. This change may be necessary in order to have the labeled or unlabeled edges in appropriate positions in the row so that we may then perform the changes described in the following cases.

Figure 24. Case 1, shifting transpositions.

Case 2 Since $(j,j+1)(j,j+1) = 1$ in S_n, we can insert $(j,j+1)(j,j+1)$ into our sequence of transpositions, and the resulting sequence is still be a representation of the identity. We note that this requires unlabeled edges in the row in order to perform this change in a single step, and the resulting row has two additional labeled edges. We see in Figure 25 that it is possible to preserve adjacency when making this change, so this is consistent with the definition of a G-homotopy. In the image of the new row, there is one new edge, traversed twice in opposite directions.

Figure 25. Case 2 and Case 3, inserting or removing pairs of transpositions.

Case 3 Similarly, we can remove $(j, j+1)(j, j+1)$ from the sequence of transpositions. This has the reverse effect of the change in case 2, so it also preserves adjacency in the G-homotopy grid. The effect is to remove a single edge, traversed twice, in opposite directions, from the image of one row to the next.

Case 4 Since $(j, j+1)(k, k+1) = (k, k+1)(j, j+1)$ if $|j-k| \geq 2$, we can commute two disjoint transpositions in the sequence of edge labels. Again, this requires at least one unlabeled edge in an appropriate location in the row so that we may preserve adjacency in the grid while making the change. In Γ_{B_n}, the image of one row includes two adjacent edges of a 4-cycle, and the image of the next row contains the other two edges of the cycle. The images of the two rows differ by a 4-cycle, which is contractible to a single vertex, so this change is also consistent with the definition of a G-homotopy.

Figure 26. Case 4, commuting disjoint transpositions.

Case 5 A transposition in S_n is its own inverse, so we can express the relation in case

4 as $(j, j+1) = (k, k+1)(j, j+1)(k, k+1)$ if $|j-k| \geq 2$. Thus we can replace one transposition $(j, j+1)$ with a sequence of three transpositions $(k, k+1)(j, j+1)(k, k+1)$, where $(j, j+1)$ and $(k, k+1)$ are disjoint, assuming we have unlabeled edges where necessary to preserve adjacency when the change is made. In Γ_{B_n}, the image of one row contains one edge of a 4-cycle, and the image of the next row contains the other three edges. Again, the images of the two rows differ by a 4-cycle. This is equivalent to first inserting $(k, k+1)(k, k+1)$ on one side of $(j, j+1)$, then commuting one copy of $(k, k+1)$ with $(j, j+1)$, although this may be done simultaneously from one row to the next in the grid. We may also reverse this, and replace $(k, k+1)(j, j+1)(k, k+1)$ with $(j, j+1)$ when the transpositions are disjoint, or equivalently, commute and then remove transpositions.

Figure 27. Case 5, replacing one transposition with three transpositions.

Case 6 We may extend the previous type of change, and replace a sequence of transpositions $(j_1, j_1+1)(j_2, j_2+1)\cdots(j_m, j_m+1)$ with $(k, k+1)(j_1, j_1+1)(j_2, j_2+1)\cdots(j_m, j_m+1)(k, k+1)$ where $(k, k+1)$ is disjoint from each of the other transpositions in the subsequence. Similarly, this is equivalent to inserting $(k, k+1)(k, k+1)$ then commuting repeatedly, and the images of the two consecutive rows in the grid differ by m 4-cycles, one for each of the transpositions that we commuted with $(k, k+1)$.

Figure 28. Case 6, inserting a pair of transpositions and commuting with a sequence of disjoint transpostions.

We used the relations $(j, j+1)(j, j+1) = 1$ and $(j, j+1)(k, k+1) = (k, k+1)(j, j+1)$ if $|j-k| \geq 2$ in the changes described above in cases 2 through 6, but we have not yet used the third relation in our representation for S_n, $((j,j+1)(j+1,kj+2))^3 = 1$. If we attempt to use this relation in order to make a change in the grid, for example, replacing a sequence of the form $(j,j+1)(j+1,j+2)(j,j+1)$ with $(j+1,j+2)(j,j+1)(j+1,j+2)$, we see that the image of the first row includes three consecutive edges of a reduced 6-cycle associated with $(j,j+1)$ and $(j+1,j+2)$, and the image of the next row contains the other three edges. Consequently, the images of the two rows differ by a 6-cycle, which we have seen is not contractible to a single vertex. Therefore this type of change cannot occur in a G-homotopy, where the images of two consecutive rows may only differ by 3- and 4-cycles or by one or more edges traversed twice in opposite directions. It is easy to see that any change to the sequence of transpositions that we make using the third relation results in two rows whose images in Γ_{B_n} differ by a 6-cycle. Consequently, the changes that we are allowed to make from row to row in a G-homotopy grid may only rely on the first two relations.

Possible changes are therefore limited to shifting labeled edges, inserting or deleting a pair of transpositons $(j,j+1)(j,j+1)$, commuting disjoint transpositions, or a combination of these changes. Inserting or deleting pairs of transpositions does not affect the parity of the total number of transpositions of each type contained in consecutive rows of the grid, and shifting labeled edges or commuting transpositions does not affect the number of each type of transposition from row to row. Thus the first row of the G-homotopy grid from C_1 to C_2 must contain an odd number of edges labeled with each of $(i-1,i)$ and $(i,i+1)$, and consequently the 6-cycle C_1 must also be associated with $(i-1,i)$ and $(i,i+1)$, which completes the proof of the lemma. $\square$

6.2. G-homotopic 6-cycles Differ by a Sequence of Transpositions

Association with the same pair of transpositions is a necessary condition for G-homotopy of 6-cycles, but it turns out not to be sufficient. In order to guarantee that two 6-cycles, C_1 and C_2, are G-homotopic, they must also differ by a sequence of simple transpositions, $\tau_1\tau_2...\tau_k$, where each τ_j is disjoint from $(i-1,i)$ and $(i,i+1)$. That is, if we multiply each of the six permutations in C_1 by the same sequence $\tau_1\tau_2\cdots\tau_k$, the result is precisely the six permutations in C_2. To indicate this relationship, we write $C_2 = C_1\tau_1\tau_2\cdots\tau_k$.

Theorem 6.2. *Let C_1 and C_2 be two distinct reduced 6-cycles in Γ_{B_n}. Then $C_1 \simeq_G C_2$ iff there exists an integer $k \geq 1$ such that $C_2 = C_1\tau_1\ldots\tau_k$ where C_1 and C_2 are both associated to $(i-1,i)$ and $(i,i+1)$ for some i, $2 \leq i \leq n-1$, and the τ_j are simple transpositions in S_n that are disjoint from $(i-1,i)$ and $(i,i+1)$.*

Proof. The first part of the proof is constructive: assuming $C_2 = C_1\tau_1\ldots\tau_k$, we construct a G-homotopy from C_1 to C_2 whose image is a sequence of 6-cycles connected by 4-cycles. In the second part of the proof we assume $C_1 \simeq_G C_2$, which means there is a path P such that $C_1PC_2^{-1}P^{-1} \simeq_G \sigma_0$, where the base permutation σ_0 is a permutation in C_1. The edges of P correspond to simple transpositions in S_n, and the product of these transpositions is a permutation in S_n. We show that this permutation can be written using only transpositions that are disjoint from the pair $(i-1,i)$ and $(i,i+1)$ associated to C_1 and C_2.

Figure 29. A G-homotopy from C_1 to $C_2 = C_1\tau_1\tau_2\tau_3$.

Constructing a G-homotopy from $\mathbf{C_1}$ to $\mathbf{C_2 = C_1\tau_1\tau_2\tau_3}$. First, suppose $C_2 =$

$C_1\tau_1 \ldots \tau_k$ as defined in the statement of the theorem, and suppose $C_1 = \sigma_0 - \sigma_1 - \sigma_2 - \sigma_3 - \sigma_4 - \sigma_5 - \sigma_0$. We can construct a G-homotopy from C_1 to PC_2P^{-1} where $P = \sigma_0 - \sigma_0\tau_1 - \sigma_0\tau_1\tau_2 - \cdots - \sigma_0\tau_1 \ldots \tau_k$. The G-homotopy grid has $k+1$ rows and $2k+7$ columns, and Figure 29 is an example of such a grid in the case where $k = 3$. The first row of the grid corresponds to C_1, with $k+1$ copies of the base permutation σ_0, then $\sigma_1 - \sigma_2 - \sigma_3 - \sigma_4 - \sigma_5$, followed by $k+1$ copies of σ_0. We note that only the six edges in the row corresponding to C_1 are labeled (colored) with an alternating sequence of $(i-1, i)$ and $(i, i+1)$, and all other edges in the row are unlabeled. In row 2, we leave the first and last copies of σ_0 unchanged, and we multiply each of the other permutations in the row by τ_1. This is equivalent to inserting two copies of the transposition τ_1 at the beginning the sequence of transpositions associated to C_1, then commuting one of the copies of τ_1 with the six copies of $(i-1, i)$ and $(i, i+1)$ as described in case 6 in the previous section. The image of this new row is the loop $\sigma_0 - \sigma_0\tau_1 - \sigma_1\tau_1 - \sigma_2\tau_1 - \sigma_3\tau_1 - \sigma_4\tau_1 - \sigma_5\tau_1 - \sigma_0\tau_1 - \sigma_0$, which is also the 6-cycle $C_1\tau_1$ connected to σ_0 by an edge associated with τ_1.

In each of the remaining rows, row $j+1$ for $2 \leq j \leq k$, we obtain the new row by leaving the first j permutations and the last j permutations unchanged, and multiplying the other permutations in the previous row by τ_j. Again, this is a change like that in case 6, so the result of these changes is a valid G-homotopy. Thus the images of the rows is the sequence of 6-cycles $C_1\tau_1, C_1\tau_1\tau_2, \ldots, C_1\tau_1 \ldots \tau_k$, connected to the base permutation σ_0 by a subpath of P. In particular, the permutations in the last row are

$$\sigma_0 - \sigma_0\tau_1 - \sigma_0\tau_1\tau_2 - \cdots - \sigma_0\tau_1 \ldots \tau_k - \sigma_1\tau_1 \ldots \tau_k - \cdots - \sigma_5\tau_1 \ldots \tau_k - \sigma_0\tau_1 \ldots \tau_k - \cdots - \sigma_0\tau_1 - \sigma_0$$

which corresponds to the loop PC_2P^{-1}.

If P is a shortest path from σ_0 in C_1 to C_2, then $k+1$ is the least number of rows that a G-homotopy from C_1 to C_2 could have. The image of the G-homotopy that we have constructed is a sequence of $k+1$ 6-cycles, $C_1, C_1\tau_1, C_1\tau_1\tau_2, \ldots, C_1\tau_1 \ldots \tau_k$, each corresponding to $(i-1,i),(i,i+1)$ for the same i. Each pair of consecutive 6-cycles is connected by six edges corresponding to the same transposition, as shown in Figure 30. Therefore, if σ is a permutation in C_1, then $\sigma\tau_1\tau_2 \ldots \tau_k$ is a permutation in C_2.

Figure 30. The image of a G-homotopy from C_1 to C_2.

Assume $\mathbf{C_1 \simeq_G C_2}$. For the second part of the proof, we now assume $C_1 \simeq_G C_2$ and $C_2 = \gamma_0 - \gamma_1 - \gamma_2 - \gamma_3 - \gamma_4 - \gamma_5 - \gamma_0$. Then by Lemma 6.1, C_1 and C_2 are both associated with the same pair of transpositions $(i-1,i)$ and $(i,i+1)$. Clearly, $\gamma_0 - \gamma_1 - \gamma_2 - \gamma_3 - \gamma_4 - \gamma_5 - \gamma_0$ is G-homotopic to $\gamma_1 - \gamma_2 - \gamma_3 - \gamma_4 - \gamma_5 - \gamma_0 - \gamma_1$, so as with C_1, we may choose to label the permutations in C_2 so that the edge incident to γ_0 and γ_1 is associated with $(i-1,i)$. For this part of the proof, it is useful to express this relationship as $C_1PC_2^{-1}P^{-1} \simeq_G \sigma_0$, where P is a shortest path from σ_0 in C_1 to C_2, satisfying the condition that the first edge traversed in C_2 is associated with $(i-1,i)$.

The first row in a G-homotopy grid corresponds to $C_1PC_2^{-1}P^{-1}$ and the last row is repeated copies of the base permutation σ_0. Recall that we label an edge in the grid if it corresponds to a simple transposition, otherwise it is left unlabeled. Therefore the last row has no labeled edges. This means we must use our permissible operations of commuting disjoint permutations, inserting or deleting a pair of consecutive copies of the same transposition, or shifting labeled edges, as we move from one row to the next in the grid. Our ultimate goal is to eventually be able to remove **all** of the transpositions in $C_1PC_2^{-1}P^{-1}$ by the last row of the grid.

Suppose we insert a pair of transpositions, $(j, j+1)(j, j+1)$, at some step in our process. By the end of our process, each of these transpositions has been removed as a part of a pair of transpositions. If they were removed together as a single pair, then they did not assist us in removing other pairs of transpositions, so it was not necessary to insert them at all. If they were removed separately, each paired with another transposition of the same type, then whatever commuting that was done to put them into position next to their new partners could have been done in the opposite direction by the partner transpositions. This would make a single consecutive pair out of these new partners, which we could then remove. In this case as well, we could have constructed a G-homotopy between $C_1PC_2^{-1}P^{-1}$ and the trivial loop σ_0 without inserting the new pair of transpositions. Thus we must be able to achieve our goal solely by shifting labeled edges and commuting and removing transpositions.

A shortest path $\mathbf{P}$ from $\mathbf{C_1}$ to $\mathbf{C_2}$. Now we consider the types of transpositions that may be associated with the edges of a shortest path from C_1 to C_2. The path P from σ_0 in C_1 to C_2 consists of edges corresponding to simple transpositions $\tau_1, \tau_2, \ldots, \tau_k$ in S_n, therefore the sequence of labels on the edges in the first row of the grid is

$$(i-1,i)(i,i+1)\ldots(i,i+1)\tau_1\tau_2\ldots\tau_k(i,i+1)(i-1,i)\ldots(i-1,i)\tau_k\tau_{k-1}\ldots\tau_1.$$

We refer to P as a sequence of transpositions, the transpositions associated with the edges in P, although we recognize that by commuting disjoint permutations in the sequence we are changing the edges traversed and creating a new path, albeit a path that begins and ends at the same vertices (permutations) as P, and the product of the new sequence of transpositions still represents the same permutation in S_n. If we can show that P consists of transpositions disjoint from $(i-1,i)$ and $(i,i+1)$, then we could commute the transpositions in P with the transpositions in C_2^{-1}, resulting in the sequence

$$(i-1,i)(i,i+1)\ldots(i,i+1)(i,i+1)(i-1,i)\ldots(i-1,i)\tau_1\tau_2\ldots\tau_k\tau_k\tau_{k-1}\ldots\tau_1.$$

We can then remove pairs of the same transposition, one pair at a time, until all transpositions have been removed.

P doesn't contain $(\mathbf{i-2,i-1})$ or $(\mathbf{i+1,i+2})$. We show that P cannot contain any copies of either $(i-2,i-1)$ or $(i+1,i+2)$. Suppose we have a transposition $(i-2,i-1)$ in P, and if P contains more than one copy of this transposition, then we consider the last copy in P. There are an odd number of copies of $(i-1,i)$ between the last copy of $(i-2,i-1)$ in P and the first copy in P^{-1}; three from C_2^{-1} and an even number, if any, from P and P^{-1} between the two copies of $(i-2,i-1)$. We can't commute $(i-2,i-1)$ with $(i-1,i)$, so we won't be able to remove all of the odd number of copies of $(i-1,i)$ to allow us to pair the copies of $(i-2,i-1)$ from P and P^{-1} for cancellation. Therefore we must be able to pair the copy of $(i-2,i-1)$ from P with a second copy that is also in P. But if we can commute and remove these transpositions within P as part of the G-homotopy, then we could have commuted and removed these transpositions within P to get a shorter path from C_1 to C_2 with the same endpoints, which contradicts our assumption that P is a shortest path. Therefore P cannot contain any copies of $(i-2,i-1)$, and by a similar argument, any copies of $(i+1,i+2)$.

Suppose now that P contains copies of $(i-1,i)$ or $(i,i+1)$. We show that this contradicts the assumption that P is a shortest path from σ_0 in C_1 to C_2, satisfying the condition that the first edge traversed in C_2 is associated with $(i-1,i)$. We consider the subsequence of transpositions $(i-1,i)$ and $(i,i+1)$ in P and show that each of the possible types of subsequences results in a contradiction of our assumption about P. When we have eliminated all possibilities, then we may conclude that all transpositions in P are disjoint from $(i-1, i)$ and $(i, i+1)$. First, we show that a subsequence can't have consecutive copies of either $(i-1, i)$ or $(i, i+1)$. Next, we show that a subsequence must be shorter than six transpositions. We then show a G-homotopy is not possible if the alternating subsequence is of odd length. Finally, for subsequences of length two or four, we demonstrate a shorter path satisfying the conditions for P.

Subsequence of $(\mathbf{i-1, i})$, $(\mathbf{i, i+1})$ alternates. Suppose that the subsequence in P consisting of all transpositions $(i-1, i)$ and $(i, i+1)$ contains a consecutive pair the same transposition. Transpositions $(i-1, i)$ and $(i, i+1)$ commute with all other transpositions in P (except, of course, with each other), so if there are consecutive repetitions of one of the transpositions in the subsequence, we can commute the transpositions within P so they are next to one another and then remove the pair to obtain a shorter path from C_1 to C_2 with the same endpoints, contradicting our assumption that P is a shortest path. Therefore any subsequence must alternate between $(i-1, i)$ and $(i, i+1)$.

Length of subsequence $<$ 6. Next we consider what types of alternating subsequences of $(i-1, i)$ and $(i, i+1)$ are possible in our G-homotopy. If we have an alternating subsequence of length six or longer, then we can commute transpositions from P until we have six of these transpositions in consecutive order. This sequence of six transpositions is a representation of the identity in S_n, so we now have a path from C_1 to C_2 that contains a 6-cycle. The same path without the 6-cycle yields a shorter path from C_1 to C_2 with the same endpoints, contradicting our assumption that P is a shortest path. Therefore we only need to consider subsequences of length less than six.

Odd subsequences. For subsequences of odd length, Figure 31 illustrates the possible subsequences of transpositions $(i-1, i)$ and $(i, i+1)$ in the first row of the grid, using (12) and (23) as a specific example. This includes the six transpositions from each of

C_1 and C_2, as well as those from P and P^{-1}. Even though (12) doesn't commute with (23), (12) and (23) commute with any other transposition in P, thus we only need to consider the subsequence when deciding which pairs of (12) or (23) may be removed. In each case, we see that we can remove a number of pairs of transpositions, equal to the number of transpostions in the subsequence in P. However, we are not able to remove all transpositions in the entire subsequence, contradicting our assumption that $C_1 \simeq_G C_2$.

Figure 31. Cancelling transpositions in $C_1PC_2^{-1}P^{-1}$ when subsequences in P are of odd length.

Even subsequences. Now suppose that the alternating subsequence in P is of length two or four. To illustrate this we consider the case where the subsequence is $(i-1,i),(i,i+1)$. We can commute these transpositions with the other transpositions from P to obtain a new sequence

$$(i-1,i),(i,i+1),\tau_{j_1},\tau_{j_2},\ldots,\tau_{j_{k-2}}$$

with $(i-1,i),(i,i+1)$ and the rest of the transpositions in the same relative order as they were in P. The first vertex in P is associated with the permutation is σ_0, and without loss of generality we can identify the last permutation in the path, which is a permutation in C_2, to be γ_0. Our reordered sequence of transpositions also corresponds to a path from σ_0 to γ_0 in Γ_{B_n}, thus we may write

$$\begin{aligned}\gamma_0 &= \sigma_0(i-1,i)(i,i+1)\tau_{j_1}\tau_{j_2}\ldots\tau_{j_{k-2}}\\ &= \sigma_2\tau_{j_1}\tau_{j_2}\ldots\tau_{j_{k-2}}.\end{aligned}$$

Consequently, we also have

$$\begin{aligned}
\gamma_1 &= \gamma_0(i-1,i)\\
&= \sigma_2\tau_{j_1}\tau_{j_2}\ldots\tau_{j_{k-2}}(i-1,i)\\
&= \sigma_2(i-1,i)\tau_{j_1}\tau_{j_2}\ldots\tau_{j_{k-2}}\\
&= \sigma_3\tau_{j_1}\tau_{j_2}\ldots\tau_{j_{k-2}}
\end{aligned}$$

and similarly

$$\begin{aligned}
\gamma_2 &= \sigma_4\tau_{j_1}\tau_{j_2}\ldots\tau_{j_{k-2}}\\
\gamma_3 &= \sigma_5\tau_{j_1}\tau_{j_2}\ldots\tau_{j_{k-2}}\\
\gamma_4 &= \sigma_0\tau_{j_1}\tau_{j_2}\ldots\tau_{j_{k-2}}\\
\gamma_5 &= \sigma_1\tau_{j_1}\tau_{j_2}\ldots\tau_{j_{k-2}}.
\end{aligned}$$

Therefore $C_2 = C_1\tau_{j_1}\tau_{j_2}\ldots\tau_{j_{k-2}}$ and we could use the sequence of edges starting at σ_0 and corresponding to $\tau_{j_1}, \tau_{j_2}, \ldots, \tau_{j_{k-2}}$ to create a shorter path from σ_0 to C_2 in Γ_{B_n}. This new path does not meet C_2 at the same permutation as does P, but it still satisfies our condition that the first edge traversed in C_2 is associated with $(i-1,i)$ (we can simply reindex the γ_j so that the new path ends at γ_0). This contradicts our assumption that P is a shortest path from σ_0 to C_2. It is easy to show that in the other three cases, where the subsequence is of length four and beginning with $(i-1,i)$, or of length two or four and beginning with $(i,i+1)$, the 6-cycles C_1 and C_2 differ by the sequence of remaining transpositions from P not in the subsequence, and we can use those transpositions to create a shorter path from σ_0 to C_2, ending at a new permutation but still satisfying our condition that the first edge traversed in C_2 is associated with $(i-1,i)$, for a contradiction. Consequently, P cannot contain any edges corresponding to $(i-1,i)$ or $(i,i+1)$.

We have shown that if $C_1 \simeq_G C_2$ and P is a shortest path from σ_0 in C_1 to C_2, then the edges of P must correspond to a sequence of simple transpositions $\tau_1, \tau_2, \ldots, \tau_k$ in S_n that are disjoint from $(i-1,i)$ and $(i,i+1)$, and without loss of generality, if we let γ_0 be the last permutation in P, then

$$\begin{aligned}\gamma_0 &= \sigma_0\tau_1\tau_2\dots\tau_k \\ \gamma_i &= \gamma_{i-1}(i-1,i) \\ &= \sigma_{i-1}(i-1,i)\tau_1\tau_2\dots\tau_k \\ &= \sigma_i\tau_1\tau_2\dots\tau_k \qquad\qquad \text{for } 1 \le i \le 5.\end{aligned}$$

Therefore $C_2 = C_1\tau_1\tau_2\dots\tau_k$ as desired.

□

6.3. Characterization of Equivalence Classes

The previous theorem stems from the definition of a G-homotopy from C_1 to C_2 and the limitations on the types of changes we are able to make from row to row in a G-homotopy grid. We can combine this theorem with our new understanding of the structure of Γ_{B_n} to make the following observations about equivalence classes of 6-cycles in the graph: We show that the horizontal and vertical 6-cycles are not G-homotopic to one another, so we may discuss horizontal and vertical equivalence classes. Next, we see that the vertical 6-cycles at different levels are in different equivalence classes. Then we demonstrate a method for counting the vertical equivalence classes at each level i of Γ_{B_n}, for $2 \le i \le n-1$. Finally we show that the number of horizontal equivalence classes in Γ_{B_n} is equal to the total number of equivalence classes in $\Gamma_{B_{n-1}}$, and show that the set of horizontal and vertical equivalence classes that we have identified constitute a minimal set of generators for $A_1^{n-3}(\Delta(\overline{B_n}))^{ab}$.

Horizontal and vertical 6-cycles are in different equivalence classes. In the construction of Γ_{B_n}, we saw that $\sigma(j) = n$ for each permutation σ at level j of the graph. Therefore, given a vertical 6-cycle C_2 that spans levels $i-1$, i, and $i+1$, we have $\sigma(i-1) = n$

for the two permutations in C_2 at level $i-1$, $\sigma(i) = n$ for the two permutations at level i, and $\sigma(i+1) = n$ for the remaining two permutations at level $i+1$. On the other hand, $\sigma(j) = n$ for all six permutations in a horizontal 6-cycle C_1 at level j in Γ_{B_n}. Furthermore, $\sigma^{-1}(n) = \tau_k\tau_{k-1}\cdots\tau_2\tau_1(j)$ for each permutation $\sigma\tau_1\tau_2\cdots\tau_k$ in $C_1\tau_1\tau_2\cdots\tau_k$, so $C_1\tau_1\tau_2\cdots\tau_k$ is also a horizontal 6-cycle. As a result, it is not possible to have $C_2 = C_1\tau_1\tau_2\ldots\tau_k$ where C_1 is a horizontal 6-cycle and C_2 is a vertical 6-cycle, and so we may consider equivalence classes of horizontal and vertical equivalence classes separately.

Vertical 6-cycles at different levels of Γ_{B_n} are in different equivalence classes. This is a direct consequence of Lemma 6.1 and the observation we made in Section 6.1 of this chapter that a vertical 6-cycle at level i, $2 \leq i \leq n-1$ is associated with transpositions $(i-1, i)$ and $(i, i+1)$.

There are $\binom{\mathbf{n-1}}{\mathbf{i}}\binom{\mathbf{i}}{\mathbf{2}}$ equivalence classes at level i of $\mathbf{\Gamma_{B_n}}$, $\mathbf{2 \leq i \leq n-1}$. Let C_1 be a vertical 6-cycle at level i of Γ_{B_n} and let σ be a permutation in C_1 at level i. While every vertical 6-cycle at level i is associated to $(i-1, i)$ and $(i, i+1)$, this does not mean they are all in the same equivalence class. We know that if C_2 is another vertical 6-cycle at level i and it is G-homotopic to C_1, then there is an integer $k \geq 1$ and simple tranpositions $\tau_j, 1 \leq j \leq k$, such that $C_2 = C_1\tau_1\tau_2\ldots\tau_k$. The τ_j need not be distinct, so we may consider any permutation generated by the set of τ_j. Therefore, the number of 6-cycles in the equivalence class containing C_1 is $(i-2)!(n-i-1)!$, the order of the subgroup of S_n generated by all simple transpositions in S_n except $(i-2, i-1), (i-1, i), (i, i+1)$, and $(i+1, i+2)$. Recall that $\Gamma_{B_{n-1}}$ is $(n-2)$-regular. When we use $\Gamma_{B_{n-1}}$ to construct $\tilde{\Gamma}_{B_n}$, the vertices in each level i, $2 \leq i \leq n-1$, are incident to two new vertical edges, thus each vertex in these levels has degree n. The graph Γ_{B_n} is n-regular, so we removed precisely one horizontal edge incident to each vetex in these levels. Removing a horizontal edge resulted in the creation of a vertical 6-cycle, therefore each vertex in level i, $2 \leq i \leq n-1$, is in precisely one vertical 6-cycle at level i. Furthermore, each of the 6-cycles in a single equivalence class contains two vertices at level i and there is a total of $(n-1)!$ permutations at level i. Thus there are $\frac{(n-1)!}{2(i-2)!(n-i-1)!} = \binom{n-1}{i}\binom{i}{2}$ equivalence classes at level i, and the total number of vertical equivalence classes in Γ_{B_n} is

$$\begin{aligned}
\textstyle\sum_{i=2}^{n-1} \binom{n-1}{i}\binom{i}{2} &= \textstyle\sum_{i=2}^{n-1} \frac{(n-1)!}{2(i-2)!(n-i-1)!} \\
&= \textstyle\sum_{i=0}^{n-3} \frac{(n-1)!}{2(i)!(n-i-3)!} \\
&= \tfrac{(n-1)(n-2)}{2} \textstyle\sum_{i=0}^{n-3} \binom{n-3}{i} \\
&= 2^{n-3}\binom{n-1}{2}.
\end{aligned}$$

The number of horizontal equivalence classes in $\boldsymbol{\Gamma_{B_n}}$ is equal to the total number of equivalence classes in $\boldsymbol{\Gamma_{B_{n-1}}}$. Recall that when we constructed Γ_{B_n} using n copies of $\Gamma_{B_{n-1}}$, no edges were removed from the top level, level n, of $\widetilde{\Gamma}_{B_n}$ in order to obtain Γ_{B_n}. Furthermore, each permutation in level n, written in single line notation, has the number n in the last position, position n. Let C be a horizontal or vertical 6-cycle in $\Gamma_{B_{n-1}}$. If we map each permutation σ in C to a permutation σ' in S_n by putting the element n at the end of the permutation, we obtain a horizontal 6-cycle, C', in level n of Γ_{B_n}. For example, we see in Figures 21 and 22 that the single 6-cycle $123-213-231-321-312-132$ in Γ_{B_3} corresponds in this way to the 6-cycle $1234-2134-2314-3214-3124-1324$ in Γ_{B_4}. We also note that these two 6-cycles are associated with the same pair of transpositions, (12) and (23). Thus there is a bijection between 6-cycles in $\Gamma_{B_{n-1}}$ and horizontal 6-cycles in level n of Γ_{B_n}, and the corresponding pairs of 6-cycles in the two graphs are each associated with the same pair of transpositions. We show that there is also a bijection between equivalence classes in $\Gamma_{B_{n-1}}$ and equivalence classes in Γ_{B_n} that contain horizontal 6-cycles in level n.

Lemma 6.3. *Let C_1 and C_2 be two distinct 6-cycles in $\Gamma_{B_{n-1}}$, then $C_1 \simeq_G C_2$ in $\Gamma_{B_{n-1}}$ if and only if $C_1' \simeq_G C_2'$ in Γ_{B_n}.*

Proof. Suppose $C_1 \simeq_G C_2$ in $\Gamma_{B_{n-1}}$. Then $C_2 = C_1\tau_1\tau_2\cdots\tau_k$ as defined in Theorem 6.2. Each of the τ_j is a simple transposition in S_{n-1}, so in particular the set of τ_j cannot include $(n-1, n)$. Therefore, if we multiply each of the permutations in C_1' by the sequence of τ_j, the entry n is always in the last position. In fact, if we have a G-homotopy from C_1 to C_2 in $\Gamma_{B_{n-1}}$, we may simply put the number n at the end of each of the permutations in the grid and the result is a G homotopy from C_1' to C_2' in Γ_{B_n} that demonstrates that $C_2' = C_1'\tau_1\tau_2\cdots\tau_k$.

Now suppose that $C_1' \simeq_G C_2'$ in Γ_{B_n}. Then $C_2' = C_1'\tau_1\tau_2\cdots\tau_k$. Recall that C_1' and C_2' are both horizontal 6-cycles in level n of Γ_{B_n}, so all permutations in both cycles have the entry n in position n. Therefore we do not need the transposition $(n-1, n)$ in our sequence of τ_j because the transpositions do not need to move the entry in the last position. This means that each of the τ_j are also simple transpositions in S_{n-1}, so we can use the same collection of transpositions to show that $C_2 = C_1\tau_1\tau_2\cdots\tau_k$, and therefore $C_1 \simeq_G C_2$ in $\Gamma_{B_{n-1}}$ as desired.

□

The bijection between 6-cycles in $\Gamma_{B_{n-1}}$ and horizontal 6-cycles in level n of Γ_{B_n}, together with the above lemma, induce a bijection between the equivalence classes of 6-cycles in $\Gamma_{B_{n-1}}$ and equivalence classes in Γ_{B_n} that contain horizontal 6-cycles in level n. Now we show that those are in fact the only horizontal equivalence classes that we need to consider. Let C_1 be a horizontal 6-cycle in level i of Γ_{B_n} for some i, $1 \leq i \leq n-1$. No vertical edges were removed from $\widetilde{\Gamma}_{B_n}$ in the construction of Γ_{B_n}, and each permutation in level i is connected to a corresponding permutation in level $i+1$ by an edge associated with $(i, i+1)$. These two permutations originated from a single permutation from S_{n-1} in $\Gamma_{B_{n-1}}$, with the element n inserted into the permutation at position i or $i+1$, respectively, to obtain two permutations in S_n. Thus we can find a 6-cycle, C_2 in level n of Γ_{B_n} such that the permutations in C_1 and C_2 originated from a single 6-cycle in $\Gamma_{B_{n-1}}$, with the element n inserted into the permutations at position i or n, respectively.

If σ in C_1 and σ' in C_2 originated from the same permutation in S_{n-1}, then $\sigma' = \sigma(i, i+1)(i+1, i+2)\cdots(n-1, n)$, and the permutations are connected by a path of vertical edges corresponding to these transpositions. Similarly

$$C_2 = C_1(i, i+1)(i+1, i+2)\cdots(n-1, n),$$

and the two 6-cycles are connected by six such paths. In $\widetilde{\Gamma}_{B_n}$, each pair of consecutive vertical paths around the 6-cycles were connected at each level by horizontal edges. For an illustration of this, we may consider the unique 6-cycles in each of levels 1 and 4 in $\widetilde{\Gamma}_{B_4}$ in Figure 21, which we see are connected by a "net" of vertical 4-cycles. In Figure 22, this net has been modified somewhat due to removing edges to obtain Γ_{B_4}, and the net now consists of vertical 4-cycles and 6-cycles. If, in Γ_{B_n}, the net consists of only 4-cycles, as in the subgraph of Γ_{B_5} shown in Figure 32, then $C_1 \simeq_G C_2$, and C_1 is contained in one of the equivalence classes we have already considered.

Figure 32. A subgraph of Γ_{B_5} showing two G-homotopic 6-cycles connected by a net of vertical 4-cycles.

On the other hand, if the net consists of both 4-cycles and 6-cycles, then $C_1 \not\simeq_G C_2$, but C_1 is G-homotopic to the concatenation of C_2 with the vertical cycles in the net. The

4-cycles are all in the equivalence class of a single vertex, and we have already counted the equivalence classes containing the vertical 6-cycles as well. Therefore C_1 is contained in the product of other equivalence classes that have already been counted. Consequently, when counting equivalence classes of horizontal 6-cycles, it is sufficient to consider equivalence classes containing horizontal 6-cycles in level n of Γ_{B_n}, which we have shown is equal to the total number of equivalence classes in $\Gamma_{B_{n-1}}$.

There are $\mathbf{2^{n-3}(n^2-5n+8)-1}$ equivalence classes of 6-cycles in $\mathbf{\Gamma_{B_n}}$. When we tally the total number of equivalence classes of 6-cycles in Γ_{B_n}, we add the number of "new" equivalence classes of vertical 6-cycles, $2^{n-3}\binom{n-1}{2}$, to the "old" equivalence classes containing horizontal 6-cycles which we have shown were counted in $\Gamma_{B_{n-1}}$. Thus by induction, there are $\sum_{k=1}^{n} 2^{k-3}\binom{k-1}{2}$ equivalence classes of 6-cycles in Γ_{B_n}. Any other cycle in Γ_{B_n} of length ≥ 8 can be expressed as the concatenation of 4-cycles and 6-cycles, so these equivalence classes generate our free group $A_1^{n-3}(\Delta(B_n))^{ab}$. Therefore our goal is to find a generating function for

$$f(n) = \sum_{k=1}^{n} 2^{k-3}\binom{k-1}{2}.$$

To accomplish this, we begin by rewriting this sum so that the binomial also depends on n, which reverses the order of the terms if we expand the sum. Furthermore, for values $k=0$ and $k \geq n$, the terms vanish, so we can sum over all values $k \geq 0$. Our new function is

$$f(n) = \sum_{k\geq 0} 2^{n-k-3}\binom{n-k-1}{2}.$$

Using the Snake Oil Method [16], we multiply the above sum by x^n, and sum over values of n. This is our generating function, $F(x)$, and we would like to find a closed formula for the coefficients of $F(x)$.

$$F(x) = \sum_{n} x^n \sum_{k\geq 0} 2^{n-k-3} \binom{n-k-1}{2}.$$

Next we reverse the order of summation, and then we adjust the exponent for x and for 2 to match the form in the binomial. We then replace $n-k-1$ with a dummy variable r, and sum over r.

$$\begin{aligned}
F(x) &= \textstyle\sum_{k\geq 0} \sum_{n} 2^{n-k-3} \binom{n-k-1}{2} x^n \\
&= \textstyle\sum_{k\geq 0} 2^{-2} x^{k+1} \sum_{n} 2^{n-k-1} \binom{n-k-1}{2} x^{n-k-1} \\
&= \textstyle\sum_{k\geq 0} \frac{1}{4} x^{k+1} \sum_{r} \binom{r}{2} (2x)^r \\
&= \textstyle\sum_{k\geq 0} \frac{1}{4} x^{k+1} \frac{(2x)^2}{(1-2x)^3} \\
&= \textstyle\sum_{k\geq 0} \frac{x^{k+3}}{(1-2x)^3} \\
&= \textstyle\frac{x^3}{(1-2x)^3} \sum_{k\geq 0} x^k \\
&= \textstyle\frac{x^3}{(1-2x)^3(1-x)}.
\end{aligned}$$

Using standard partial fraction decomposition techniques, we obtain the result that the coefficient of x^n in $F(x)$ is $2^{n-3}(n^2-5n+8)-1$. This is the total number of equivalence classes of 6-cycles in Γ_{B_n}, and the number of a minimal set of generators of $A_1^{n-3}(\Delta(\overline{B_n}))^{ab}$. We have also recovered a formula for the first Betti number of $M_{n,3}$, which Björner and Lovász [6] showed give us a lower bound for the complexity of the k-equal problem described in the introduction of this dissertation.

6.4. Future Directions

The results in Chapters 5 and 6, related to the Boolean lattice, were possible in large part due to the construction of the graph related to a direct product of lattices that was developed in Chapter 4. Our new understanding of the structure of $\Gamma_{B_n} = \Gamma^{n-3}_{max}(\Delta(\overline{B_n}))$, gained from viewing B_n as the direct product $B_{n-1} \times \mathbf{2}$, was indispensable in proving our results. However, the arguments also relied heavily on the properties of S_n, specifically our ability to limit our investigation to the equivalence classes of reduced 6-cycles in the graph and to determine that types of changes that are possible from row to row in a G-homotopy grid. One path for future work beyond this dissertation is to investigate direct products of other lattices to see if the structure of the associated graph can aid in obtaining results for lattices other than B_n.

Recall that in the introduction to this dissertation, we briefly described Maurer and Malle's early work in discrete homotopy theory. Maurer [13] investigated matroid basis graphs, where each vertex corresponds to a basis of a matroid, and two vertices are adjacent if the corresponding bases differ by a single exchange. He defined a homotopy relation on paths that is equivalent to our G-homotopy of graph maps. He proved that Γ is a matroid basis graph if and only if

1. it is connected,

2. each common neighborhood subgraph is a square, a pyramid, or an octohedron,

3. in every leveling each common neighbor subgraph meets the Positioning Condition (which we do not define here), and

4. for some v_0 the neighborhood subgraph $N(v_0)$ is the line graph of a bipartite graph.

He then conjectured that conditions 3 and 4 can be replaced by a new condition, which is that $A_1^G(\Gamma)$ is trivial. Since Maurer published his conjecture, Malle [12] characterized graphs with a trivial string fundamental group (which corresponds to our $A_1^G(\Gamma, v)$) as the collection of graphs where each cycle has a pseudoplanar net. Malle's characterization might give us new insight into graphs with a trivial G-homotopy group, and it has the potential to be valuable tool in proving Maurer's conjecture.

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